Bondal-Orlov's reconstruction theorem in noncommutative projective geometry (arXiv:2411.07813)

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Introduction: Reconstruction Problems

k : alg clo field.

Question

When can we renconstruct a scheme from
its (derived) category of coherent sheaves ?

Theorem (Gabriel '62)

X, Y : noeth schs.
Then,

$$\mathrm{Coh}(X) \simeq \mathrm{Coh}(Y) \Rightarrow X \simeq Y.$$

Theorem (Bondal-Orlov '01)

X, Y : sm proj vars over k.
Assume that the canonical bundles of X, Y are (anti-)ample.
Then,

$$D^b(\mathrm{Coh}(X)) \simeq D^b(\mathrm{Coh}(Y)) \Rightarrow X \simeq Y.$$

Introduction: NC Proj Geometry

- $A = \bigoplus_{i \in \mathbb{N}} A_i$: locally fin (i.e. $\dim_k A_i < \infty$) noeth $\mathbb{N}$-gr k-alg.
- $\mathrm{qgr}(A) = \mathrm{gr}(A)/\mathrm{tor}(A)$: quotient cat of $\mathrm{gr}(A)$ by $\mathrm{tor}(A)$.
  - ▷ $\mathrm{obj}(\mathrm{qgr}(A)) = \mathrm{obj}(\mathrm{gr}(A))$,
  - ▷ $\mathrm{Hom}_{\mathrm{qgr}(A)}(\pi_A(M), \pi_A(N)) = \varinjlim_n \mathrm{Hom}_{\mathrm{gr}(A)}(M_{\geq n}, N_{\geq n})$,

    where $\pi_A : \mathrm{gr}(A) \to \mathrm{qgr}(A)$ is the projection.

Definition (Artin-Zhang '94)

The noncommutative (NC) projective scheme associated to A is $\mathrm{qgr}(A)$.

Theorem (M, rough version)

Under appropriate conditions,
Bondal-Orlov's reconstruction theorem holds for NC proj schs.

Canonical Bimodules & Ampleness on Abelian Cats

$\mathcal{C}$: abelian cat, $\mathcal{O}$: obj in $\mathcal{C}$, $F : \mathcal{C} \xrightarrow{\simeq} \mathcal{C}$: autoeq.

Definition (Mori-Ueyama '21, Artin-Zhang '94)

1. F is an canonical bimodule on $\mathcal{C}$ if
$$F[n] : D^b(\mathcal{C}) \to D^b(\mathcal{C})$$
is a Serre functor for ${}^{\exists}n \in \mathbb{Z}$.
2. $(\mathcal{O}, F)$ is ample if
   - (i) ${}^{\forall}M \in \mathcal{C}$, ${}^{\exists}$ epimor $\varphi : \bigoplus_{i=1}^{r} F^{-\ell_i}(\mathcal{O}) \twoheadrightarrow M \quad (\ell_1, \cdots, \ell_r \in \mathbb{N})$,
   - (ii) ${}^{\forall}$ epimor $f : M \twoheadrightarrow N$, ${}^{\exists}m_0 \in \mathbb{N}$ s.t. the natural mor
$$\mathrm{Hom}_{\mathcal{C}}(F^{-m}(\mathcal{O}), M) \to \mathrm{Hom}_{\mathcal{C}}(F^{-m}(\mathcal{O}), N),$$
is surj for ${}^{\forall}m \geq m_0$.

$(\mathcal{O}, F)$ is anti-ample if $(\mathcal{O}, F^{-1})$ is an ample.

Remark

L : line bundle on X.

L : amp canonical bdl $\Leftrightarrow$ $- \otimes L$: canonical bimod & $(\mathcal{O}_X, - \otimes L)$: amp.

Dualizing Complexes of NC Graded Algebras

Definition (Yekutieli '92)

A dualizing complex (dc) of A is a cpx $R \in D^b(\mathrm{Gr}(A^{en}))$ s.t.

1. R has fin inj dim & fin gen cohomology over A & A^{op},
2. The functor

$$\mathbf{R}\,\mathrm{Hom}_A(-, R) : D^b(\mathrm{gr}(A)) \to D^b(\mathrm{gr}(A^{\mathrm{op}}))$$

   is an equiv with inverse $\mathbf{R}\,\mathrm{Hom}_{A^{\mathrm{op}}}(-, R)$.

Moreover, R is balanced if $\mathbf{R}\Gamma_{\mathfrak{m}_A}(R) \simeq \mathbf{R}\Gamma_{\mathfrak{m}_{A^{\mathrm{op}}}}(R) \simeq A'$ (graded k-dual).

Remark

A has a balanced dc & $\mathrm{qgr}(A)$ has a can bimod
$\Rightarrow \pi_A(- \otimes_A H^{-(n+1)}(R))$: can bimod of $\mathrm{qgr}(A)$.

Main Theorem

A, B : loc fin noeth $\mathbb{N}$-gr k-algs with balanced dc.

Theorem (M)

Assume that $\mathrm{qgr}(A), \mathrm{qgr}(B)$ have canonical bimodules K_A, K_B.
If $(\pi_A(A), K_A), (\pi_B(B), K_B)$ are (anti-)ample, then

$$\boldsymbol{D^b(\mathrm{qgr}(A)) \simeq D^b(\mathrm{qgr}(B)) \Rightarrow \mathrm{qgr}(A) \simeq \mathrm{qgr}(B)}.$$

<u>Remark</u>

- Main theorem $\Rightarrow$ Original Bondal-Orlov reconstruction.
- To prove thm, we need to show
  1. $\forall G : D^b(\mathrm{qgr}(A)) \xrightarrow{\simeq} D^b(\mathrm{qgr}(B))$, G : Fourier-Mukai functor.
  2. The canonical algs of A and B are iso, i.e.

$$\bigoplus_{m \in \mathbb{N}} H^0(\mathrm{qgr}(A), K_A^m(\pi_A(A))) \simeq \bigoplus_{m \in \mathbb{N}} H^0(\mathrm{qgr}(B), K_B^m(\pi_B(B))).$$

AS Regular Algebras

A : connected (i.e. $A_0 = k$) fin gen $\mathbb{N}$-gr k-alg.
$k = A/A_{>0}$ is regarded as an A-mod.

Definition (Artin-Schelter '87)

A is Artin-Schelter (AS) regular if

1. $\mathrm{gl.dim}(A) < \infty$,
2. $\{\dim_k A_i\}_{i \in \mathbb{N}}$ has poly growth.
3. A is Gorenstein, i.e. $\mathrm{Ext}^i_A(k, A) \simeq \begin{cases} 0 & (i \neq d) \\ k & (i = d) \end{cases}$.

Example

- A : commutative AS reg alg $\Leftrightarrow$ A : polynomial algebra.
- skew polynomial rings, Sklyanin algebras, Feigin-Odesskii's elliptic algebras, etc.

An Application of Main Thoeorem

Corollary (M)

Let A, B be noetherian AS regular algebras.
Then,

$$\boldsymbol{D^b(\mathrm{qgr}(A)) \simeq D^b(\mathrm{qgr}(B)) \Rightarrow \mathrm{qgr}(A) \simeq \mathrm{qgr}(B)}.$$

Remark

- Corollary holds for locally fin ver of AS regular algs.
- **Even when proving the connected case,**
  we need to consider locally fin ver of AS regular algs !
  In detail, the notion of quasi-Veronese algebras are important.

Theorem (M)

A, B : loc fin noeth $\mathbb{N}$-gr k-algs w/ balanced dc.
Assume that $\text{qgr}(A), \text{qgr}(B)$ have can bimods K_A, K_B.

If $(\pi_A(A), K_A), (\pi_B(B), K_B)$ are (anti-)ample, then

$$\mathbf{D^b(qgr(\mathit{A})) \simeq D^b(qgr(\mathit{B})) \Rightarrow qgr(\mathit{A}) \simeq qgr(\mathit{B})}.$$

Corollary (M)

A, B : noeth AS regular algs.
Then,

$$\mathbf{D^b(qgr(\mathit{A})) \simeq D^b(qgr(\mathit{B})) \Rightarrow qgr(\mathit{A}) \simeq qgr(\mathit{B})}.$$

Thank you for your attention.