An explicit construction of derived moduli stacks of Harder-Narasimhan filtrations

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§1 Introduction

$F : Sch^{op} \to (Groupoids)$

$F(id_S) = id_{F(S)}$

$F(g \circ f) = F(f) \circ F(g)$

$S \longmapsto Hom(-, S)$

$Sch \hookrightarrow Fun(Sch^{op}, Sets)$

$F(id_S) \cong id_{F(S)}$

$F(g \circ f) \cong F(f) \circ F(g)$

$\hookrightarrow Fun(Sch^{op}, Groupoids)$

$\hookrightarrow Pse\text{-}Fun(Sch^{op}, Groupoids)$

(= Categories fibered in groupoids)

Groupoids $\subseteq$ Category

cats whose mors are iso.

stack = pseudo-funct satisfying the glueing condition

Motivation of derived algebraic geometry

k : alg-closed fld
$\alpha \in \mathbb{Q}[t]$
X : Proj sch / k of finite type
$\mathcal{O}_X(1)$: ample sheaf

$$\mathrm{Coh}_\alpha(X) : (k\text{-sch})^{op} \longrightarrow (\text{Groupoids})$$

$$S \longmapsto \mathrm{Coh}_\alpha(X)(S) := \begin{cases} \cdot\ \underline{\text{obj}} & E : S\text{-flat fam of coh sh on } X \text{ with Hilb poly } \alpha. \\ \cdot\ \underline{\text{mor}} & \varphi : E \to F \text{ iso of sheaves} \end{cases}$$

$$(f : T \longrightarrow S) \longmapsto \mathrm{Coh}_\alpha(X)(f) := f^*(-)$$

(pull back of $\mathcal{O}_X$-mods)

<u>Rmk</u> $\mathrm{Coh}_\alpha(X)$ is an Algebraic (Artin) stack,

i.e. $\exists\ f : \underset{\text{sch}}{U} \longrightarrow \mathrm{Coh}_\alpha(X)$ representable smooth, surj mor

(atlas)

<u>Note</u> $\cdot$ $f : U \to \mathrm{Coh}_\alpha(X)$: rep $\underset{\text{def}}{\Longleftrightarrow}$ $U \times_{\mathrm{Coh}_\alpha(X)} V$: scheme

$$\begin{tikzcd} U \times_{\mathrm{Coh}_\alpha(X)} V \arrow[r, "f'"] \arrow[d] \arrow[dr, phantom, "\square"] & V \arrow[d, "\forall g"] \\ U \arrow[r, "f"'] & \mathrm{Coh}_\alpha(X) \end{tikzcd}$$

• $f : U \to V$ smooth mor of k-schemes

then,

$$0 \to T_{U/V} \to T_U \to f^*T_V \to 0$$

(exact)

In other words,

$$f^*T_V \underset{\substack{\uparrow \\ \text{quasi-iso}}}{\simeq} [T_{U/V} \to T_U] \in D^{[-1,0]}(U)$$

• $f : U \to Coh_\alpha(X)$ Rep smooth surj mor

$$\underline{f^* T_{Coh_\alpha(X)}} := [T_{U/Coh_\alpha(X)} \to T_U] \in D^{[-1,0]}(U)$$

where $T_{U/Coh_\alpha(X)} := \Delta^*(T_{U \times_{Coh(X)} U/U})$

$$\begin{tikzcd}
U \arrow[r, "\Delta"] \arrow[dr, "id_U"'] & U \times_{Coh(X)} U \arrow[r, "f'"] \arrow[d] \arrow[dr, phantom, "\square"] & U \arrow[d, "f"] \\
& U \arrow[r, "f"] & Coh_\alpha(X)
\end{tikzcd}$$

- $\mathrm{Coh}_\alpha(X)$ is smooth $\overset{}{\underset{\text{def}}{\Longleftrightarrow}}$ U ; smooth

$[E] \in \mathrm{Coh}_\alpha(X)$: k-point (i.e. E : coh on X)
$x \in U$: a lift of $[E]$.

$T_E \mathrm{Coh}_\alpha(X) := [\, T_{U/\mathrm{Coh}_\alpha(X)} \otimes k(x) \to T_U \otimes k(x) \,]$
$\chi(T_E \mathrm{Coh}_\alpha(X)) := -\dim T_{U/\mathrm{Coh}(X)} \otimes k(x) + \dim T_U \otimes k(x)$

<u>Rmk</u> $\chi(T_E \mathrm{Coh}_\alpha(X))$: locally const
$\Rightarrow \mathrm{Coh}_\alpha(X)$: smooth

Moreover, we can show

$$T_E \mathrm{Coh}_\alpha(X) \underset{\text{quasi-iso}}{\cong} \tau_{\le 0}\left(\mathbb{R}\mathrm{Hom}(E,E)[1]\right)$$

<u>Rmk</u> X : curve $\Rightarrow \mathrm{Coh}_\alpha(X)$: smooth
$\dim X \ge 2 \Rightarrow \mathrm{Coh}_\alpha(X)$: not sm in general.

$\Big((\because)$ X : curve $\Rightarrow \chi(T_E \mathrm{Coh}_\alpha(X))$: const
$\dim(X) \ge 2 \Rightarrow \chi(T_E \mathrm{Coh}_\alpha(X))$ may be jump $\Big)$

Hidden smoothness philosophy (Deligne, Drinfeld, Kontsevich)

$\mathrm{Coh}_\alpha(X)$ should be just a "truncation" of suitable geometric object (derived moduli) $\mathbb{R}\mathrm{Coh}_\alpha(X)$ whose tangent cpx at $[E]$ is $\mathbb{R}\mathrm{Hom}(E, E)[1]$ $(=: T_E \mathbb{R}\mathrm{Coh})$

- X : sm var
  X locally looks like $\mathrm{Spec}(\mathrm{Sym}(T_{X,x}^\vee))$

analogy

$\mathbb{R}\mathrm{Coh}_\alpha(X)$ should locally looks like $\mathrm{Spec}(\mathrm{Sym}(T_E\mathbb{R}\mathrm{Coh}^\vee))$.

complex of vct. sps.

Differential graded (dg) scheme (stack)

$X = (X^0, \mathcal{O}_X^\bullet)$

X^0 : scheme (stack)

$\mathcal{O}_X^\bullet$: a sheaf of commutative differential graded alg (cdga) on X^0.

(graded alg + chain cpx + Leipnitz rule)

s.t. • $\mathcal{O}_X^0 = \mathcal{O}_{X^0}$

• $\mathcal{O}_X^i$: quasi-coh $\mathcal{O}_{X^0}$-mod

Previous works

- Derived moduli of semistable sheaves
  (Behrend, Ciocan-Fontanine, Hwang and Rose)
- Derived Quot schemes
  (Kapranov, Ciocan-Fontanine)
- Derived Hilb schemes
  (Kapranov, Ciocan-Fontanine)

Main Result

There exists derived moduli of HN-filts as dg-stacks, which satisfies Hidden smoothness philosophy.

Moreover, there exists ①, ②:

① Derived moduli of filt gr mods as dg-stacks

② Open embeddings of (non-derived) moduli of HN-filts into moduli of filt gr mods.

Rmk

- To obtain derived moduli of HN-filts, ①, ② are essential.
- We can construct them explicitly.

§2 Derived moduli of filtered graded modules

k : alg. closed fld of $\operatorname{ch} k = 0$

A : unital graded k-alg s.t. $A_0 = k$

(eg.) X : proj var
$A = \bigoplus_{i \geq 0} H^0(X, \mathcal{O}_X(i))$.

$\mathfrak{m} := A_{\geq 1}$

$V = \bigoplus_{i=p}^{q} V_i$: fin dim $[p,q]$-gr k-mod $(0 \leq p \leq q)$

$0 = V^0 \subseteq V^1 \subseteq \cdots \subseteq V^s = V$

: fitration of sub gr k-mod of V

$\underline{\dim}(V^i) := (\dim_k V^i_p, \cdots, \dim_k V^i_q)$ ← dimension vct

$$L^i := \operatorname{Hom}_{k\text{-gr}}(\mathfrak{m}^{\otimes i}, \operatorname{End}_k^{fil}(V^\bullet)) \quad (i \geq 0)$$

$$= \operatorname{Hom}_{k\text{-gr}, fil}(\mathfrak{m}^{\otimes i} \otimes_k V^\bullet, V^\bullet)$$

, where

$$\operatorname{End}_k^{fil}(V^\bullet) = \left\{ f : V^\bullet \to V^\bullet \;\middle|\; \begin{array}{l} f : k\text{-linear} \\ \& \text{ preserving} \\ \text{the filt of } V \end{array} \right\}$$

$$= \bigoplus_{j \in \mathbb{Z}} \operatorname{End}_k^{fil}(V^\bullet)_j$$

$$L := \bigoplus_{i \geq 0} L^i$$

Memo : $L^i = \mathrm{Hom}_{k\text{-gr}}(m^{\otimes i}, \mathrm{End}^f_k(V^\bullet))$
$= \mathrm{Hom}_{k\text{-gr, fil}}(m^{\otimes i} \otimes_k V^\bullet, V^\bullet)$

Lem $L = \bigoplus L^i$ has a structure of differential graded Lie alg (dgLa)
(= graded Liealg + chain cpx + Leipniz rule)

$d^i : L^i \to L^{i+1}$ defined by multiplicity of A
$[\cdot,\cdot] : L^i \times L^j \to L^{i+j}$ defined by composition of $\mathrm{End}^f_k(V^\bullet)$

Eg
- $d^1 : L^1 \longrightarrow L^2$
$(d^1\mu)(a_1, a_2) = \mu(a_1 a_2)$

- $[-,-] : L^1 \times L^1 \to L^2$
$[\mu,\mu'](a_1, a_2) = -\mu(a_1)\circ\mu'(a_2) - \mu'(a_1)\circ\mu(a_2)$

Maurer-Cartan elements

$$MC(L) := \{ \mu \in L^1 \mid \underbrace{d\mu + \tfrac{1}{2}[\mu,\mu] = 0}_{\text{MC-equation}} \}$$

$\mu \in MC(L) \Leftrightarrow \mu(a_1 a_2) = \mu(a_1)\circ\mu(a_2)$
$\Leftrightarrow$ μ defines (non unital) filt gr m-act on $V^\bullet$
$\Leftrightarrow$ μ defines (unital) filt gr A-act on $V^\bullet$

To construct derived moduli of filt A-gr mods, we define derived moduli of filt gr A-act on $V^\bullet$ as dg-schs.

$X = (X^0, \mathcal{O}_X^\bullet)$
X^0 : sch
$\mathcal{O}_X^\bullet$: sheaf of cdga on X^0

Def (M.)

- $Y^0 := L^1 = \mathrm{Hom}_{k\text{-gr, fil}}(M \otimes_k V^\bullet, V^\bullet)$

  $\mathcal{L}^i := L^1 \times_k L^i \quad (i \geq 2)$ triv vct bdls on L^1

  $\mathcal{L} := \bigoplus_{i \geq 2} \mathcal{L}^i$

- $\mathcal{O}_Y^\bullet := \mathrm{Sym}_{\mathcal{O}_{Y^0}}(\mathcal{L}[1]^\vee) \in D^{[-\infty, 0]}(Y^0)$

  $Y := (Y^0, \mathcal{O}_Y^\bullet)$ derived moduli of filt gr A-act on $V^\bullet$.

Rmk

- The differential of $\mathcal{O}_Y^\bullet$ is induced by d, $[\cdot\ \cdot]$ & a sect def by MC-eq on $\mathcal{L}^2$.

- $\tau_0(Y) := \mathrm{Spec}(H^0(\mathcal{O}_Y^\bullet)) \simeq MC(L)$
  $\simeq$ "moduli of filt gr A-act on V"

truncation

<u>Rmk</u> $G := GL_{gr}(V) = \prod_{i=p}^{q} GL(V_i)$

$P := \prod_{i=p}^{q} P_i \subseteq G$ the filts of $V^\bullet$ preserved

we have an act of P on $L = \bigoplus L^i$
from the act of P on $\mathrm{End}_K^f(V^\bullet)$.

Def (M.)

$\tilde{Y} := (\tilde{Y}^0, \mathcal{O}_{\tilde{Y}}^\bullet)$

, where $\tilde{Y}^0 := [Y^0/P]$ and $\mathcal{O}_{\tilde{Y}}^\bullet \in D^{[-\infty, 0]}(\tilde{Y}^0)$

($\mathcal{O}_Y^\bullet$ is P-equivariant, so $\mathcal{O}_Y^\bullet$ descends to $\tilde{Y}^0$.)

we call $\tilde{Y}$ derived moduli of filt gr A-mod with the dim vect of $V^\bullet$.

Thm 1 $\tilde{Y}$ Satisfies Hidden sm philosophy.

i.e.,

- $T_0(\tilde{Y}) := \mathrm{Spec}(H^0(\mathcal{O}_{\tilde{Y}}^\bullet)) \cong [MC(L)/P]$
  $\cong$ 「moduli of filt gr A-mods with dim vect of $V^\bullet$」

- $T_{M^\bullet}\tilde{Y} \underset{\text{qis}}{\cong} \mathbb{R}\mathrm{Hom}_{A\text{-}gr,\,filt}(M^\bullet, M^\bullet)[1]$

  for any filt gr A-mod $M^\bullet$.

§3 Embedding the moduli stack of HN-filt into the moduli stack of filt gr A-mods.

$$\begin{cases} X : \text{projective variety} / k \\ \mathcal{O}_X(1) : \text{ample sheaf on } X. \\ A := \bigoplus_{i \geq 0} \Gamma(X, \mathcal{O}_X(i)) \end{cases}$$

Def

- E : torsion free sheaf on X.

Harder-Narasimhan (HN)-filt of E (w.r.t $\mathcal{O}_X(1)$)

$$0 = E^0 \subseteq E^1 \subseteq \cdots \subseteq E^s = E$$

s.t.
- E^i/E^{i-1} : semistable for $\forall i$
- $p^{\mathrm{red}}(E^1/E^0) > \cdots\cdots > p^{\mathrm{red}}(E^s/E^{s-1})$

Rmk Any torsion free sheaf has a unique HN-filtration.

「the HN-type of E」 $:= (P(E^1/E^0), \cdots, P(E^s/E^{s-1}))$

Note $\alpha_1, \cdots, \alpha_s \in \mathbb{Q}[t]$, $\alpha := \alpha_1 + \cdots + \alpha_s$

$M^\bullet$: filt $[p, q]$-gr A-mod

the type of $M^\bullet$ is $(\alpha_1, \cdots, \alpha_s)$

$\overset{\text{def}}{\Longleftrightarrow}$ $\underline{\dim}(M^i/M^{i-1}) = (\alpha_i(p), \cdots, \alpha_i(q))$

- $\exists\, \mathrm{Coh}^{HN}_{(\alpha_1 \cdots \alpha_s)}(X)$

$$S \longmapsto \left\{ \begin{array}{l} S\text{-flat fam } \mathcal{E}^\bullet \text{ of HN-} \\ \text{filts of type } (\alpha_1, \cdots, \alpha_s) \end{array} \right\}$$

- $\exists\, \mathrm{Mod}^{[p,q]}_{(\alpha_1, \cdots, \alpha_s)}(A)$

$$S \longmapsto \left\{ \begin{array}{l} S\text{-flat fam } \mathcal{M}^\bullet \text{ of filt } [p,q]\text{-gr} \\ A\text{-mods of type } (\alpha_1, \cdots, \alpha_s) \end{array} \right\}$$

For $p \gg 0$, we can define

$$\Gamma^{fil}_{[p,q]} : \exists\, \mathrm{Coh}^{HN}_{(\alpha_1, \cdots, \alpha_s)}(X) \longrightarrow \exists\, \mathrm{Mod}^{[p,q]}_{(\alpha_1, \cdots, \alpha_s)}(A)$$

$$0 \subseteq \mathcal{E}^1 \subseteq \cdots \subseteq \mathcal{E}^s \longmapsto 0 \subseteq \Gamma_{[p,q]}(\mathcal{E}^1) \subseteq \cdots \subseteq \Gamma_{[p,q]}(\mathcal{E}^s)$$

, where $\Gamma_{[p,q]}(\mathcal{E}^i) = \bigoplus_{j=p}^{q} \pi_{S*}(\mathcal{E}^i(j))$.

Thm 2 $q \gg p \gg 0$, $\Gamma^{fil}_{[p,q]}$: open immersion

Idea of Prf

$\Gamma^{fil}_{[p,q]}$: open imm $\Leftrightarrow \left\{ \begin{array}{l} \text{① monomorphism} \quad \left(\begin{array}{c}\text{relatively}\\ \text{easy}\end{array}\right) \\ \text{② étale mor} \end{array} \right.$

To prove ②, we show

$$\mathrm{Ext}^i_{fil}(\mathcal{E}^\bullet, \mathcal{E}^\bullet) = \mathrm{Ext}^i_{A\text{-}gr, fil}(\Gamma^{fil}_{[p,q]}(\mathcal{E}^\bullet), \Gamma^{fil}_{[p,q]}(\mathcal{E}^\bullet))$$

By using this, we compare the corresp deformation functors.

Def

- (λ, M) : a pair of
$$\begin{cases} M : [p,q]\text{-gr } k\text{-mod} \\ \lambda : A \otimes_k M \to M \quad \text{hom of gr } k\text{-mod.} \end{cases}$$

- $\theta_p, \theta_q \in \mathbb{Z}$ stability parameter.

$0 \neq {}^\forall N \subseteq M$, s.t. $\lambda(A \otimes N) \subseteq N$, $\dim N_p + \dim N_q \neq 0$.

$$\mu_{(\theta_p, \theta_q)}(N) := \frac{\theta_p \dim N_p + \theta_q \dim N_q}{\dim N_p + \dim N_q}$$

HN-filt of (M, λ) w.r.t. (θ_p, θ_q).

$0 = M^0 \subseteq M^1 \subseteq M^2 \subseteq \cdots \subseteq M^s = M$

s.t. $(M^i / M^{i-1}, \lambda)$: semistable w.r.t (θ_p, θ_q)

$\mu(M^1/M^0) > \cdots > \mu(M^s/M^{s-1})$.

Prop For $q \gg p' \gg p \gg 0$, we have

$$\mathrm{Im}(\Gamma^{fil}_{[p,q]}) = \left\{ 0 = M^0 \subseteq M^1 \subseteq \cdots \subseteq M^s \;\middle|\; \begin{array}{l} M^i : \text{gen in deg } p \\ M^\bullet : \text{HN-filt} \\ \quad \text{w.r.t } (\alpha(q), -\alpha(p)) \\ M^\bullet_{\geq p'} : \text{HN-filt} \\ \quad \text{w.r.t } (\alpha(q), -\alpha(p')) \end{array} \right\} (☆)$$

Memo $\alpha := \alpha_1 + \cdots + \alpha_s$

§4 Derived enhancement of moduli of HN-filts

- we take p, p', q so that Prop holds.

$0 = V^0 \subsetneq V^1 \subsetneq \cdots \subsetneq V^s = V$. filt $[p,q]$-gr k-mod of type $(\alpha_1, \cdots, \alpha_s)$.

$$L' \underset{\text{open}}{\supseteq} L'' := \left\{ \lambda : A \otimes_k V^\bullet \to V^\bullet \ (\text{unital}) \;\middle|\; \begin{array}{l} \lambda_p : A \otimes_k V^\bullet_p \to V^\bullet \ \text{surj} \\ (\lambda, V^\bullet) : \text{HN-filt w.r.t } (\alpha(q), -\alpha(p')) \\ (\lambda, V^\bullet_{\geq p'}) : \text{HN-filt w.r.t } (\alpha(q), -\alpha(p')) \end{array} \right\}$$

then,

$$\mathrm{Im}\left(\Gamma^{fil}_{[p,q]}\right) = [L'' \cap MC(L)/P] \underset{\text{open}}{\subseteq} [MC(L)/P] \cong \mathcal{J}\mathrm{Mod}^{[p,q]}_{(\alpha_1,\cdots,\alpha_s)}(A)$$

Def (M.) $Y^{0\prime} := L''$ $\left([Y^{0\prime}/P] \underset{\text{open}}{\subseteq} [Y^0/P] = \tilde{Y}^0 \right)$

$$\mathbb{R}\mathcal{J}\mathrm{Coh}^{HN}_{(\alpha_1,\cdots,\alpha_s)}(X) := \left([Y^{0\prime}/P],\ \mathcal{O}^\bullet_{\tilde{Y}}\big|_{[Y^{0\prime}/P]} \right)$$

we call this derived moduli of HN-filt of type $(\alpha_1, \cdots, \alpha_s)$ on X.

Thm3 (Main Result)

$\mathbb{R}\mathfrak{Z}\mathrm{Coh}^{HN}_{(\alpha_1,\cdots,\alpha_s)}(X)$ satisfies Hidden sm philosophy.

i.e.,

- $t_0\,\mathbb{R}\mathfrak{Z}\mathrm{Coh}^{HN}_{(\alpha_1,\cdots,\alpha_s)}(X) \cong \mathfrak{Z}\mathrm{Coh}^{HN}_{(\alpha_1,\cdots,\alpha_s)}(X)$

- $[E^\bullet] \in \mathfrak{Z}\mathrm{Coh}^{HN}_{(\alpha_1,\cdots,\alpha_s)}(X)$: k-pt

$$T_{E^\bullet}\,\mathbb{R}\mathfrak{Z}\mathrm{Coh}^{HN}_{(\alpha_1,\cdots,\alpha_s)}(X) \underset{\text{quasi-iso}}{\simeq} \mathbb{R}\mathrm{Hom}_{filt}(E^\bullet, E^\bullet)[1]$$

($(\because)$ (Idea)
Using Thm 1, 2 and Prop.)

Thank you for listening !