# Bondal-Orlov's reconstruction theorem in noncommutative projective geometry

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July 5, 2025

# Introduction: Reconstruction Problems

$k$ : alg clo field.

## Question

When can we renconstruct a scheme from **its (derived) category of coherent sheaves ?**

## Theorem (Gabriel '62)

$X, Y$ : noeth schs.
Then,

$$\mathrm{Coh}(X) \simeq \mathrm{Coh}(Y) \Rightarrow X \simeq Y.$$

## Theorem (Bondal-Orlov '01)

$X, Y$ : sm proj vars over $k$.
Assume that the canonical bundles of $X, Y$ are (anti-)ample.
Then,

$$D^b(\mathrm{Coh}(X)) \simeq D^b(\mathrm{Coh}(Y)) \Rightarrow X \simeq Y.$$

# Introduction : motivation of NC Proj geometry

- $R = \bigoplus_{i \in \mathbb{N}} R_i$ : commutative $\mathbb{N}$-gr $k$-alg
- $\mathrm{gr}(R)$ : category of fin gen $\mathbb{Z}$-gr $R$-mods
- $\mathrm{qgr}(R) = \mathrm{gr}(R)/\,\mathrm{tor}(R)$ : quotient category.
  - ▷ $\mathrm{obj}(\mathrm{qgr}(R)) = \mathrm{obj}(\mathrm{gr}(R))$,
  - ▷ $\mathrm{Hom}_{\mathrm{qgr}(R)}(\pi_R(M), \pi_R(N)) = \varinjlim_n \mathrm{Hom}_{\mathrm{gr}(R)}(M_{\geq n}, N_{\geq n})$.

    ($\pi_R : \mathrm{gr}(R) \to \mathrm{qgr}(R)$ : the projection)

**Theorem (Serre '55)**

Assume that $R$ is generated by $R_1$ as an $R_0$-alg.
Then,

$$\mathrm{Coh}(\mathrm{Proj}(R)) \simeq \mathrm{qgr}(R).$$

By [Serre '55] + [Gabriel '62], we can think that

**qgr($R$) is essential in projective algebraic geometry !**

# Introduction: NC Proj Geometry

- $A = \bigoplus_{i \in \mathbb{N}} A_i$ : locally fin noeth $\mathbb{N}$-gr $k$-alg.
- $\mathrm{gr}(A)$ : cat of fin gen $\mathbb{Z}$-gr right $A$-mods.
- $\mathrm{qgr}(A) = \mathrm{gr}(A)/\mathrm{tor}(A)$ : quotient cat.

### Definition (Artin-Zhang '94)

We call
$\mathrm{qgr}(A)$ the noncommutative projective (NC) scheme associated with $A$.

### Theorem (M, rough version)

Under appropriate conditions,
**Bondal-Orlov's reconstruction theorem holds for NC proj schs**.

**Key notions**

Canonical bundles, their (anti-)ampleness and dualizing complexes
in NC world.

# Canonical Bimodules in the Theory of Abelian Categories

$\mathcal{C}$ : $k$-linear abelian cat, $F : \mathcal{C} \to \mathcal{C}$ : autoequiv.

**Definition (Mori-Ueyama '21)**

$F$ is a canonical bimodule on $\mathcal{C}$ if ${}^{\exists} n \in \mathbb{Z}$ s.t.

$$F[n] : D^b(\mathcal{C}) \to D^b(\mathcal{C})$$

is a Serre functor, i.e. $\mathrm{Hom}_{D^b(\mathcal{C})}(M, N) \simeq \mathrm{Hom}_{D^b(\mathcal{C})}(N, F[n](M))^\vee$.

**E.g.**

$\omega_X$ : can sheaf of a proj var $X$.

1. $X$ : sm $\Rightarrow - \otimes_{\mathcal{O}_X} \omega_X$ : can bimod on $\mathrm{Coh}(X)$ & $n = \dim(X)$.
2. $X$ : Calabi-Yau (i.e. sm & $\omega_X \simeq \mathcal{O}_X$)
   $\Leftrightarrow \mathrm{id}_{\mathrm{Coh}(X)}$ is a can bimod of $\mathrm{Coh}(X)$.

# Ampleness in the Theory of Abelian Categories

$\mathcal{O}$ : object in $\mathcal{C}$, $F : \mathcal{C} \to \mathcal{C}$ : autoequiv.

## Definition (Artin-Zhang '94)

A pair $(\mathcal{O}, F)$ is ample if

1. $^{\forall} M \in \mathcal{C}$, $^{\exists}$ an epimor $\varphi : \bigoplus_{i=1}^{r} F^{-\ell_i}(\mathcal{O}) \twoheadrightarrow M \quad (\ell_1, \cdots, \ell_r \in \mathbb{N})$.
2. $^{\forall}$ epimor $f : M \to N$, $^{\forall} m \gg 0$,

$$\begin{array}{ccc} & & F^{-m}(\mathcal{O}) \\ & \overset{\exists g}{\swarrow} & \downarrow \\ M & \xrightarrow{\ f\ } & N \end{array} \quad : \text{ commutative}$$

A pair $(\mathcal{O}, F)$ is anti-ample if $(\mathcal{O}, F^{-1})$ is ample.

**E.g.**

$L$ : an invertible sheaf on a sm proj var $X$.

- $L$ is ample on $X \Leftrightarrow (\mathcal{O}_X, - \otimes_{\mathcal{O}_X} L)$ is ample.

# Dualizing Complexes of NC Graded Algebras

## Definition (Yekutieli '92)

A dualizing complex (dc) of $A$ is a cpx $R \in D^b(\mathrm{Gr}(A^{en}))$ s.t.

1. $R$ has fin inj dim & fin gen cohomology over $A$ & $A^{\mathrm{op}}$,
2. The functor

$$\mathbf{R}\operatorname{Hom}_A(-, R) : D^b(\mathrm{gr}(A)) \to D^b(\mathrm{gr}(A^{\mathrm{op}}))$$

   is an equiv with inverse $\mathbf{R}\operatorname{Hom}_{A^{\mathrm{op}}}(-, R)$.

Moreover, $R$ is balanced if $\mathbf{R}\Gamma_{\mathfrak{m}_A}(R) \simeq \mathbf{R}\Gamma_{\mathfrak{m}_{A^{\mathrm{op}}}}(R) \simeq A'$ (graded $k$-dual).

**Rmk**

$A$ has a balanced dc & $\mathrm{qgr}(A)$ has a can bimod
$\Rightarrow \pi_A(- \otimes_A H^{-(d+1)}(R))$ : can bimod of $\mathrm{qgr}(A)$ $(d = \mathrm{gl.dim}(\mathrm{qgr}(A)))$.

# Main Theorem

$A, B$ : loc fin noeth $\mathbb{N}$-gr $k$-algs w/ balanced dc $R_A, R_B$.

### Theorem (M)

Assume that $\mathrm{qgr}(A), \mathrm{qgr}(B)$ have canonical bimodules $K_A, K_B$.
If $(\pi_A(A), K_A), (\pi_B(B), K_B)$ are (anti-)ample, then

$$\boldsymbol{D^b(\mathrm{qgr}(A)) \simeq D^b(\mathrm{qgr}(B)) \Rightarrow \mathrm{qgr}(A) \simeq \mathrm{qgr}(B)}.$$

**Rmk**

- Main theorem $\Rightarrow$ the original Bondal-Orlov's thm.
- In the prf, showing the two claims are crucial:
  1. Equivs between $D^b(\mathrm{qgr}(A))$ & $D^b(\mathrm{qgr}(B))$ is of Fourier-Mukai type.
  2. The canonical alg of $A$

$$C_A := \bigoplus_{m \in \mathbb{N}} H^0(\mathrm{qgr}(A), K_A^m(\pi_A(A)))$$

is isomorphic to $C_B$.

# AS Regular Algebras

$A$ : a connected (i.e. $A_0 = k$) noeth $\mathbb{N}$-gr $k$-alg.
$k = A/A_{>0}$ is regarded as an $A$-mod.

**Definition (Artin-Schelter '87)**

$A$ is Artin-Schelter (AS) regular if

1. $\mathrm{gl.dim}(A) < \infty$,
2. $A$ is Gorenstein, i.e. $\mathrm{Ext}^i_A(k, A) \simeq \begin{cases} 0 & (i \neq d) \\ k & (i = d) \end{cases}$.

**Rmk**

- $A$ : commutative AS reg alg $\Leftrightarrow$ $A$ : polynomial ring.

# Examples of AS regular algebras

## Example

- 1-dim AS reg alg $\simeq k[t]$.
- 2-dim AS reg alg $\simeq k\langle x, y\rangle/(xy - qyx)$ or $k\langle x, y\rangle/(xy - yx - y^m)$.
  $(q \in k^\times, m \in \mathbb{N})$
- $k\langle x_1, \cdots, x_n\rangle/(x_i x_j - q_{ij} x_j x_i \mid 1 \le i, j \le n)$, $(q_{ij} \in k^*, q_{ii} = q_{ij} q_{ji} = 1)$.

## Example (Sklyanin '83)

Let $a, b, c \in k$.

$$S_{a,b,c} := k\langle x, y, z\rangle/(f_1, f_2, f_3),$$

$$f_1 = ayz + bzy + cx^2, \ f_2 = azx + bxz + cy^2, \ f_3 = axy + bxy + cz^2.$$

There are many more (higher-dimensional) examples such as **Feigin-Odesskii's elliptic algebras**.

# An Application of Main Thoeorem

## Corollary (M)

Let $A, B$ be AS regular algebras.
Then,
$$\mathbf{D^b(qgr(A)) \simeq D^b(qgr(B)) \Rightarrow qgr(A) \simeq qgr(B)}.$$

**Rmk**

- The cor also holds for locally fin ver of AS regular algs.
- **Even when proving the connected case,**
  **we need to consider locally fin ver of AS regular algs !**
  In detail, the notion of quasi-Veronese algebras is important.

# A Question Related to NC CY mfds

- $k[x_0, \cdots, x_n]_{(q_{ij})} := k\langle x_1, \cdots, x_n\rangle/(x_ix_j - q_{ij}x_jx_i \mid 1 \le i, j \le n)$, where $q_{ij} \in k^*, q_{ii} = q_{ij}q_{ji} = 1$.

### Theorem (Kanazawa '14, M '24)

- $(d_0, \cdots, d_n) \in \mathbb{N}^{n+1}$ s.t. $d_i \mid d_0 + \cdots + d_n (=: d)$.
- $A := k[x_0, \cdots, x_n]_{(q_{ij})}/(x_0^{d/d_0} + \cdots + x_n^{d/d_n})$ with $\deg(x_i) = d_i$.

Assume

1. $q_{ij}^{d/d_i} = q_{ij}^{d/d_j} = 1, \ {}^\forall i, j$.
2. ${}^\exists c \in k^\times$ s.t. $c^{d_j} = \prod_{i=0}^n q_{ij}, \ {}^\forall j$.

Then, $\mathrm{qgr}(A)$ is CY, i.e. $\mathrm{qgr}(A)$ has a trivial cannonical bimod.

### Question

$A_1, A_2$ : $\mathbb{N}$-gr algs which satisfy the assumptions of the above thm.

$$D^b(\mathrm{qgr}(A_1)) \simeq D^b(\mathrm{qgr}(A_2)) \Rightarrow \mathrm{qgr}(A_1) \simeq \mathrm{qgr}(A_2) \ ?$$

### Theorem (M)

$A, B$ : loc fin noeth $\mathbb{N}$-gr $k$-algs w/ balanced dc.
Assume that $\mathrm{qgr}(A), \mathrm{qgr}(B)$ have can bimods $K_A, K_B$.

If $(\pi_A(A), K_A), (\pi_B(B), K_B)$ are (anti-)ample, then

$$\boldsymbol{D^b(\mathrm{qgr}(A)) \simeq D^b(\mathrm{qgr}(B)) \Rightarrow \mathrm{qgr}(A) \simeq \mathrm{qgr}(B)}.$$

### Corollary (M)

$A, B$ : noeth AS regular algs.
Then,

$$\boldsymbol{D^b(\mathrm{qgr}(A)) \simeq D^b(\mathrm{qgr}(B)) \Rightarrow \mathrm{qgr}(A) \simeq \mathrm{qgr}(B)}.$$

Thank you for your attention.