Bondal – Orlov reconstruction theorem in NC proj geometry.

① Intro (reconstruction problems and NC proj geometry)

② Main theorem and necessary notions

③ An application to the study of AS regular algebras.

④ Derived autoequivalence groups of some NC $\mathbb{P}^2$'s
(as long as time permits)

Notation

We work over a field k.

For any k-mod M, $DM = \mathrm{Hom}_k(M, k)$

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1. Intro

1.1 Reconstruction Problem

Q. when can we reconstruct a scheme from its abelian or derived category of Coh sheaves?

Thm (Gabriel '62)

X, Y : Noether schs

$$\mathrm{Coh}(X) \simeq \mathrm{Coh}(Y) \Rightarrow X \cong Y$$

Thm (Bondal-Orlov '01)

X, Y : Smooth proj vars

Assume the canonical bdls ω_X and ω_Y are (anti) ample.

$$D^b\mathrm{Coh}(X) \simeq D^b\mathrm{Coh}(Y) \Rightarrow X \cong Y.$$

Rmk It's enough to assume that either ω_X or ω_Y is (anti) amp. $\lrcorner$

1.2 NC proj geometry

- $R = \bigoplus_{i \in \mathbb{N}} R_i$ commutative f.g. $\mathbb{N}$-gr k-alg w/ $R_0 = k$.
- $\mathrm{gr}(R)$: category of f.g. $\mathbb{Z}$-gr R-mods.
- $\mathrm{tor}(R)$: full sub category of $\mathrm{gr}(R)$ consisting of tor mods.

Def we define the quotient category $\mathrm{qgr}(R)$ as follows :

- $\mathrm{Obj}(\mathrm{qgr}(R)) = \mathrm{Obj}(\mathrm{gr}(R))$
- $\mathrm{Hom}_{\mathrm{qgr}(R)}(\pi_R(M), \pi_R(N)) = \varinjlim_n \mathrm{Hom}_{\mathrm{gr}(R)}(M_{\geq n}, N_{\geq n})$

($\pi_R : \mathrm{gr}(R) \longrightarrow \mathrm{qgr}(R)$: natural quotient funct)

Rmk $\pi_R(M) \cong \pi_R(N)$ in $\mathrm{qgr}(R) \Longleftrightarrow M_{\geq n} \cong N_{\geq n}$ in $\mathrm{gr}(R)$ for $n \gg 0$

Thm (Serre '55)

Assume R is gen in deg 1.

$$\mathrm{Coh}(\mathrm{Proj}(R)) \cong \mathrm{qgr}(R)$$

Slogan (From [Serre '55] + [Gabriel '62])

$\mathrm{qgr}(R)$ is essential in Proj alg geom.

$\square$

- $A = \bigoplus_{i \in \mathbb{N}} A_i$ locally finite (i.e. $\dim_k A_i < \infty$ for $\forall i$) noether $\mathbb{N}$-gr k-alg. (not necessarily commutative)
- $\mathrm{gr}(A)$: category of f.g. $\mathbb{Z}$-gr right A-mods.
- $\mathrm{qgr}(A)$: quotient category.

Def (Artin-Zhang '94)

- We call $\mathrm{qgr}(A)$ the NC proj sch associated to A.
- We often write $\mathcal{O}_A$ for $\pi_A(A) \in \mathrm{qgr}(A)$.

Thm (M, rough version)

Under appropriate conditions BO reconstruction holds for NC proj schs.

Key Notions

Canonical bundles, (anti) ampleness in NC world.

dualizing complexes.

2. Main thm

2.1 Necessary notions

$\mathcal{C}$: k-linear abelian cat

$F : \mathcal{C} \to \mathcal{C}$ auto equivalence

<u>Def</u> (Mori-Ueyama '21)

F is called a canonical bimodule if $\exists n \in \mathbb{Z}$ s.t.

$$\tilde{F} := F \circ [n] : D^b(\mathcal{C}) \to D^b(\mathcal{C})$$

is a Serre functor, i.e. $\mathrm{Hom}(M, N) \cong D\,\mathrm{Hom}(N, \tilde{F}(M))$ for $\forall M, N \in D^b(\mathcal{C})$.

<u>E.g.</u> ω_X : can bdle of sm proj sch X

Then, $- \otimes_{\mathcal{O}_X} \omega_X$: canonical bimod on $\mathrm{Coh}(X)$.

and $n = \dim X$. $\lrcorner$

$\mathcal{O}$: object in $\mathcal{C}$.

<u>Def</u> (Artin - Zhang '94)

- A pair $(\mathcal{O}, F)$ is ample if

  ① $\forall M \in \mathcal{C}$, $\exists \varphi : \bigoplus_{i=1}^{r} F^{-l_i}(\mathcal{O}) \twoheadrightarrow M$ $\quad (l_i \in \mathbb{N})$

  ② $\forall f : M \twoheadrightarrow N$, $\forall m \gg 0$

$$\begin{tikzcd} & F^{-m}(\mathcal{O}) \arrow[ld, dashed, "\exists g"'] \arrow[d, "\forall h"] \\ M \arrow[r, "f"'] & N \end{tikzcd}$$

- A pair $(\mathcal{O}, F)$ is called anti-ample if $(\mathcal{O}, F^{-1})$ is ample.

<u>E.g</u> $L \in \mathrm{Pic}(X)$ on proj sch X.

L : ample $\Longleftrightarrow$ $(\mathcal{O}_X, - \otimes_{\mathcal{O}_X} L)$: ample

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A : locally fin noeth $\mathbb{N}$-gr k-alg.

Def (Yekutieli '92)

- A dualizing complex of A is an obj $R \in D^b(\mathrm{Gr}(A^{en}))$ s.t.

  ① R has fin. inj dim & f.g. cohomology / A & A^{op}

  ② $\mathbb{R}\mathrm{Hom}_A(-, R) : D^b(\mathrm{gr}(A))^{op} \longrightarrow D^b(\mathrm{gr}(A^{op}))$

  is equiv with inverse $\mathbb{R}\mathrm{Hom}_{A^{op}}(-, R)$

- R is balanced if $\mathbb{R}\Gamma_{\mathfrak{m}_A}(R) \cong \mathbb{R}\Gamma_{\mathfrak{m}_{A^{op}}}(R) \cong DA$

Rmk 1

- $A^{en} = A \otimes_k A^{op}$, $\mathrm{Gr}(A^{en})$: the cat of graded A-bimods.
- $\Gamma_{\mathfrak{m}_A}(-) := \varinjlim_n \mathrm{Hom}_A(A/A_{\geq n}, -)$

Rmk 2 If A has a balanced dualizing complex and $\mathrm{qgr}(A)$ has a canonical bimod

$\Rightarrow$ $\pi_A(- \otimes_A H^{-d-1}(R))$: can bimod of $\mathrm{qgr}(A)$,

where $d := \mathrm{gl.dim}(\mathrm{qgr}(A))$.

2.2 Main thm (precise statement)

- A, B : locally finite noeth $\mathbb{N}$-gr k-algs w/ balanced dualizing complex R_A, R_B
- Assume $\mathrm{qgr}(A)$, $\mathrm{qgr}(B)$ have canonical bimods ω_A, ω_B

Thm (M)

Assume $(\mathcal{O}_A, \omega_A)$ and $(\mathcal{O}_B, \omega_B)$ are (anti) amp

$$D^b\mathrm{qgr}(A) \cong D^b\mathrm{qgr}(B) \Rightarrow \mathrm{qgr}(A) \cong \mathrm{qgr}(B)$$

Rmk
- Main thm + [Serre '55] + [Gabriel '62] $\Rightarrow$ Original BO.
- The proof is different from the original one

Reason It is difficult to deal with point-like obj's in NC geometry

Def X : smooth proj var of dim n.

$P \in D^b\mathrm{Coh}(X)$ is point-like if

① $S_X(P) = P[n]$ (S_X : Serre funct of X),

② $\mathrm{Hom}(P, P)$: fld, $\mathrm{Hom}(P, P[i]) = 0$ ($\forall i < 0$)

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key Lemma

$F : D^b \mathrm{qgr}(A) \longrightarrow D^b \mathrm{qgr}(B)$ any equivalence.

Then, F is of Fourier-Mukai type i.e.,

$\exists\ \mathcal{E} \in D^b \mathrm{qbigr}(A^{op} \otimes_k B)$ s.t.

$$F(M) \cong \pi_B (R\rho_A(M) \overset{\mathbb{L}}{\otimes}_A R\rho_{A^{op}\otimes_k B}(\mathcal{E}))$$

for $\forall M \in D^b \mathrm{qgr}(A)$

Rmk
- $A^{op} \otimes_k B$ is a bigraded k-alg with $\deg(a \otimes b) = (\deg a, \deg b)$
- $\mathrm{qbigr}(A^{op} \otimes_k B) := \mathrm{bigr}(A^{op} \otimes_k B) / \mathrm{tor}(A^{op} \otimes_k B)$
- "$\overset{\mathbb{L}}{\otimes}_A$" is derived tensor product of dg-mods over dg-cat $\mathcal{A}$ associated to A.
- ρ_A, $\rho_{A^{op}\otimes_k B}$: the right adjoints of π_A, $\pi_{A^{op}\otimes_k B}$, respectively.

By using key lemma, we prove that the canonical alg $C_A^{\pm}$ of A

$$C_A^{+} := \bigoplus_{m \geq 0} \mathrm{Hom}_{\mathrm{qgr}(A)} (\mathcal{O}_A, \omega_A^{m}(\mathcal{O}_A))$$

$$C_A^{-} := \text{——}\,//\,\text{——} (\mathcal{O}_A, \omega_A^{-m}(\mathcal{O}_A))$$

are isomorphic to C_B^{+} and C_B^{-}, respectively.

Fact (AZ) $(\mathcal{O}_A, \omega_A), (\mathcal{O}_B, \omega_B)$: amp $\Longrightarrow$ $\mathrm{qgr}(C_A^{+}) \cong \mathrm{qgr}(A)$, $\mathrm{qgr}(C_B^{+}) \cong \mathrm{qgr}(B)$

——//—— : anti-amp $\Longrightarrow$ $\mathrm{qgr}(C_A^{-}) \cong \mathrm{qgr}(A)$, $\mathrm{qgr}(C_B^{-}) \cong \mathrm{qgr}(B)$

$\square$

3. Application to AS-regular algs

- A = locally finite noeth $\mathbb{N}$-gr k-alg
- $R := A_0$

Rmk
- From the iso $R \cong A/A_{>0}$, R has an A-mod str.
- We often consider the connected case $R = k$.

Def (Artin–Schelter '87, Minamoto–Mori '11)

A is Artin–Schelter (AS) regular of dim d if

① $d = \mathrm{gl.dim}(A)$, $\mathrm{gl.dim}(R) < \infty$

② A is Gorenstein, i.e.

$$\mathrm{Ext}^i_A(R, A) \cong \begin{cases} (DR)(\ell_A) & (i = d) \\ 0 & (i \neq d) \end{cases} \quad \text{in } \mathrm{gr}(A) \ \& \ \mathrm{gr}(A^{op})$$

for some $\ell_A \in \mathbb{Z}$ (Gorenstein Parameter of A)

Rmk A = connected comm AS reg alg

$\Longleftrightarrow$ A = polynomial alg. ∎

Example (Connected case.)

E.g

- 1-dim AS reg $\Longleftrightarrow$ $k[t]$
- 2-dim AS reg $\cong$ $\begin{cases} k\langle x,y\rangle/(xy-q\,yx) & (q\in k^\times) \\ k\langle x,y\rangle/(xy-yx-y^{m+1}) & (m\in\mathbb{N}) \end{cases}$
- (Skew poly alg)

$$k\langle x_1,\dots,x_n\rangle \big/ (x_i x_j - q_{ij}\, x_j x_i \mid 1\le i\le j\le n)$$

$$(q_{ij}\in k^\times,\ q_{ii}=1)$$

E.g. (Sklyanin '83)

$[a:b:c]\in\mathbb{P}^2_k\setminus\{\text{fin. pts}\}$

$$S_{a,b,c} = k\langle x,y,z\rangle \Big/ \begin{pmatrix} ayz+bzy+cx^2 \\ azx+bxz+cy^2 \\ axy+byx+cz^2 \end{pmatrix}$$

Rmk There are many more (including higher dim) examples such as Feigin - Odesskii's elliptic algs etc

Cor (M) A, B : AS regular alg.

Then,

$$D^b \mathrm{qgr}(A) \cong D^b \mathrm{qgr}(B) \Rightarrow \mathrm{qgr}(A) \cong \mathrm{qgr}(B).$$

Rmk

Even if A and B are connected AS reg algs,
we need to use non-connected AS reg algs
to prove the cor.

Reason The anti-ampleness of canonical bimod is NOT obvious.

$\square$

<u>key notions</u> The ℓ-th quasi-veronese alg $A^{[\ell]}$ of A by Mori

$$A^{[\ell]} := \bigoplus_{i \in \mathbb{N}} \begin{bmatrix} A_{\ell i} & A_{\ell i+1} & \cdots & A_{\ell i+\ell-1} \\ A_{\ell i-1} & A_{\ell i} & \cdots & A_{\ell i+\ell-2} \\ \vdots & \vdots & \ddots & \vdots \\ A_{\ell i-\ell+1} & A_{\ell i-\ell+2} & \cdots & A_{\ell i} \end{bmatrix}$$

Then, we have an equivalence $\mathrm{qgr}(A) \cong \mathrm{qgr}(A^{[\ell]})$.

<u>Fact</u> (Minamoto - Mori '11, Mori - Ueyama '21)

Suppose that $\ell = \ell_A$. Then,

- $A^{[\ell]}$ is also AS-regular
- the dualizing cpx of $A^{[\ell]}$ is $A^{[\ell]}_{\nu}(-1)[d]$ $(\nu \in \mathrm{Aut}_{\mathrm{gr}}(A^{[\ell]}))$

$\rightsquigarrow$ the funct $\pi(- \otimes_{A^{[\ell]}} A^{[\ell]}_{\nu}(-1))$ is an anti-amp canonical bimod.

$\rightsquigarrow$ we can apply the theorem.

$\square$

4. Derived autoequivalence of NC $\mathbb{P}^2$ (In Progress)

We assume $k = \bar{k}$ and $\mathrm{ch}(k) = 0$ in this section

Bondal and Orlov also proved the following in the same paper.

Thm (Bondal – Orlov '01)

X : sm proj var w/ (anti) amp canonical bdle.

$F : D^b\mathrm{Coh}(X) \to D^b\mathrm{Coh}(X)$ autoeq

Then, $\exists\, G : \mathrm{Coh}(X) \to \mathrm{Coh}(X)$ autoeq and $\exists\, m \in \mathbb{Z}$ s.t.

$$F \cong G \circ [m]$$

Question Does this hold in NC proj geometry ?

Rmk $\forall\, G : \mathrm{Coh}(X) \to \mathrm{Coh}(X)$ autoeq,

$\exists\, f : X \to X$ automor, $\exists\, L \in \mathrm{Pic}(X)$ s.t. $G(-) \cong L \otimes_{\mathcal{O}_X} f^*(-)$

As a first step for this problem, we consider low dim NC proj schs w/ (anti) amp canonical bimods!

Typical example $\mathrm{qgr}(A)$ of connected AS reg alg A generated in deg 1.

- A : 1-dim $\Rightarrow$ $A = k[t]$
- A : 2-dim $\Rightarrow$ $\mathrm{qgr}(A) \cong \mathrm{Coh}(\mathbb{P}^1)$

Because of these, we consider 3-dim AS reg algs.

Rmk A : AS reg alg of dim 3 $\Rightarrow$ $\mathrm{gl.dim}(\mathrm{qgr}(A)) = 2$.
Hence, we can regard $\mathrm{qgr}(A)$ as a NC surface.
($\otimes$ $\mathrm{gl.dim}(\mathrm{Coh}(X)) = \dim X$ for sm proj var X)

Thm (Artin-Schelter '87)
A : connected 3-dim AS reg alg generated in deg 1.
Then, A is iso to one of the following two types of algs:

① $k\langle x_1, x_2, x_3\rangle / (f_1, f_2, f_3)$ $\quad (\exists\, f_1, f_2, f_3 \in k\langle x_1, x_2, x_3\rangle_2)$

② $k\langle x_1, x_2\rangle / (g_1, g_2)$ $\quad (\exists\, g_1, g_2 \in k\langle x_1, x_2\rangle_3)$

E.g. ① type $\to$ $k[x_1, x_2, x_3]$, skew poly alg, sklyanin alg.
② type $\to$ $B = k\langle x, y\rangle / (x^2y - yx^2,\ xy^2 - y^2x)$

Def We call algebras of the first type noncommutative projective planes. (NC $\mathbb{P}^2$)

Rmk $B^{(2)} \cong k[X, Y, Z, W]/(XW - YZ)$ and $\mathrm{qgr}(B) \cong \mathrm{Coh}(\mathbb{P}^1 \times \mathbb{P}^1)$. So, algebras of the second type are often call noncommutative quadric surfaces (NC $\mathbb{P}^1 \times \mathbb{P}^1$)

Thm (Artin – Tate – Van den Bergh '90)

For any NC $\mathbb{P}^2$ A, $\exists\, i : E \hookrightarrow \mathbb{P}(V^*) \cong \mathbb{P}^2$ and $\exists\, \sigma \in \mathrm{Aut}(E)$ s.t.

- E is $\mathbb{P}^2$ or a cubic div of $\mathbb{P}^2$
- $A \cong T(V) / \langle \{ f \in V \otimes_k V \mid f(p, \sigma(p)) = 0 \ \forall p \in E \} \rangle$

  $\triangle$ We call the associated alg of $(i : E \hookrightarrow \mathbb{P}(V^*), \sigma)$

  We denote this alg by $A(E, i, \sigma)$

Rmk

- $(i_1 : E_1 \hookrightarrow \mathbb{P}(V_1^*), \sigma_1)$ and $(i_2 : E_2 \hookrightarrow \mathbb{P}(V_2^*), \sigma_2)$ give the same alg (i.e. $A(E_1, i_1, \sigma_1) \cong A(E_2, i_2, \sigma_2)$)

  $\Longrightarrow \exists\, \tau : E_1 \xrightarrow{\sim} E_2$ s.t. $\sigma_2 = \tau \circ \sigma_1 \circ \tau^{-1}$

  In particular, E & $\mathrm{ord}(\sigma)$ are uniquely determined.

- Suppose A is obtained from the pair $(i : E \hookrightarrow \mathbb{P}(V^*), \sigma)$ i.e. $A \cong A(E, i, \sigma)$

  Then,

  - $E \cong \mathbb{P}^2 \iff \mathrm{Qgr}(A) \cong \mathrm{Coh}(\mathbb{P}^2)$
  - $\mathrm{ord}(\sigma) < \infty \iff A$: finite over its center.

    Moreover, $\exists\, \mathcal{A}$: coherent sheaf of algebras on $\mathbb{P}^2$ s.t. $\mathrm{Coh}(\mathcal{A}) \cong \mathrm{Qgr}(A)$

$\square$

Thm (M, in progress)

- A : NC $\mathbb{P}^2$
- $(i : E \hookrightarrow \mathbb{P}^2, \sigma)$: a pair s.t. the associated alg $\cong A$.
  (i.e. $A(E, i, \sigma) \cong A$)

Assume that E : cubic div in $\mathbb{P}^2$ and $\mathrm{ord}(\sigma) = 2$ or 6.

Then, $\forall F : D^b \mathrm{qgr}(A) \xrightarrow{\sim} D^b \mathrm{qgr}(A)$,

$\exists G : \mathrm{qgr}(A) \xrightarrow{\sim} \mathrm{qgr}(A)$ and $\exists m \in \mathbb{Z}$ s.t.

$$F \cong G \circ [m]$$

Comment I am currently trying to prove the corresponding statement for the other orders.

Difficulty (?) It seems difficult to avoid dealing with point-like objs by using Fourier-Mukai functors.

key ingredient

Modifying the definition of point-like objects.

Flow of the Proof

1. classify modified point-like objects.
2. show the structure sheaf is sent to some shift of a locally free of rank 1 object
3. show the FM kernel is some shift of an invertible bimod. $\square$

Def $\mathcal{C}$: abelian cat of gl. dim d, $n \in \mathbb{N}$

Assume $D^b(\mathcal{C})$ has a Serre funct S.

$P \in D^b(\mathcal{C})$ is called a n-periodic point-like obj (tentative name)

if ① $S^n(P) = P[nd]$

② $\mathrm{Hom}(P, P) = k$, $\mathrm{Hom}(P, P[i]) = 0$ $(\forall i < 0)$

Rmk • 1-periodic point-like obj = classical point like obj

• 1-periodic $\subseteq$ 2-periodic $\subseteq$ 3-periodic $\subseteq$ ……

Fact (Bondal–Orlov)

$\mathcal{C} = \mathrm{Coh}(X)$, X : sm proj var of dim d w/ (anti) amp cano bdle

Then, $P \in D^b(\mathcal{C})$ is 1-periodic $\iff \left\{ \begin{array}{l} \exists x \in X, \exists m \in \mathbb{Z} \text{ s.t.} \\ P \simeq \mathcal{O}_x[m] \end{array} \right.$

Rmk S : simple in $\mathrm{Coh}(X) \iff S \cong \mathcal{O}_x$ $(\exists x \in X)$

Prop (M)

- $A = Nc\, \mathbb{P}^2$
- $(i : E \hookrightarrow \mathbb{P}^2, \sigma)$: a pair s.t. the associated alg $\cong A$.

Assume that $\mathrm{ord}(\sigma) = 2$ or 6.

Then, $P \in D^b \mathrm{qgr}(A)$ is 2-periodic point like obj

$\iff$ $\exists\, S \in \mathrm{qgr}(A)$ brick obj whose support is a single pt

$\exists\, m \in \mathbb{Z}$ s.t. $P \cong S[m]$

Rmk

- $\mathrm{Coh}(X)$ does not contain non-simple brick objs whose support is a single pt, but $\mathrm{qgr}(A)$ contains such objs
- Not all simple objs (and brick objs) in $\mathrm{qgr}(A)$ are captured as 1-periodic point-like objs.

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Thank you for your attention!