Some examples of noncommutative projective Calabi-Yau schemes

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Introduction

Aim

Construct examples of **NC proj CY schemes**.

$\downarrow$

We obtain two types of examples.

1. NC analogues of **hypersurs in weighted proj sps**
2. NC analogues of **CI in products of proj sps**

Previous Research

- NC analogues of hypersurs in (usual) proj sps (Kanazawa '14).

NC proj schemes

- $k = \overline{k}$: alg cl fld of $\mathrm{ch}(k) = 0$.
- $R = \oplus_{i \geq 0} R_i$: right noeth gr k-alg .
- $\mathrm{gr}(R)$: cat of fin gen gr right R-mods.
- $\mathrm{fdim}(R)$: cat of fin dim gr right R-mods.

Definition (NC proj schemes)

We call $(\mathbf{qgr}(R), \pi(R))$ the projective scheme of R and denote it by $\mathbf{Proj_{nc}}(R)$.

Remark

$\mathrm{qgr}(R) := \mathrm{gr}(R)/\,\mathrm{fdim}(R)$, which is the cat with

1. $\mathrm{Obj}(\mathrm{gr}(R)) = \mathrm{Obj}(\mathrm{qgr}(R))$,
2. $\mathrm{Hom}_{\mathrm{qgr}(R)}(\pi(M), \pi(N)) = \varinjlim_n \mathrm{Hom}_{\mathrm{gr}(R)}(M_{\geq n}, N_{\geq n})$,

   where $\pi : \mathrm{gr}(R) \to \mathrm{qgr}(R)$ is the projection.

Let X cpt sm var. X : CY $\overset{def}{\Longleftrightarrow} \omega_X \simeq \mathcal{O}_X$.

- A Serre functor of $\mathcal{D}$ is an equiv $S_{\mathcal{D}} : \mathcal{D} \to \mathcal{D}$ s.t. $\mathrm{Hom}_{\mathcal{D}}(E, F) \simeq \mathrm{Hom}_{\mathcal{D}}(F, \mathcal{S}_{\mathcal{D}}(E))^{\vee}$.
- gl.dim$(\mathcal{C})$:= $\mathrm{Sup}\{n \in \mathbb{Z} \mid \mathrm{Ext}^n_{\mathcal{C}}(E, F) \neq 0, {}^{\exists} E, F \in \mathrm{ob}(\mathcal{C})\}$.

Definition

$\mathrm{Proj}_{\mathrm{nc}}(R) = (\mathrm{qgr}(R), \pi(R))$ is a **proj CY *n*-scheme** if

- gl.dim $(\mathrm{qgr}(R)) = n$,
- $\mathcal{S}_{\mathrm{D}^b(\mathrm{qgr}(R))} \simeq [n]$.

Result 1

- Let $q_{ij} \in k^\times, 0 \leq i, j \leq n$.
  $k[x_0, \cdots, x_n]_{(q_{ij})} := k\langle x_0, \cdots, x_n\rangle/(x_i x_j - q_{ij} x_j x_i)_{0 \leq i,j \leq n}$.

Theorem (M)

- $(d_0, \cdots, d_n) \in \mathbb{N}^{n+1}$ *satisfying* $d_i \mid d_0 + \cdots + d_n (=: d)$.
- $A := k[x_0, \cdots, x_n]_{(q_{ij})}/(x_0^{d/d_0} + \cdots + x_n^{d/d_n})$ *with* $\deg(x_i) = d_i$.

Suppose

1. $q_{ii} = q_{ij} q_{ji} = 1, {}^\forall i, j$.
2. $q_{ij}^{d/d_i} = q_{ij}^{d/d_j} = 1, {}^\forall i, j$.

Then,
$\mathrm{Proj}_{\mathrm{nc}}(A)$ *is* **CY $(n-1)$-sch** *iff* ${}^\exists c \in k^\times$ *s.t.* $c^{d_j} = \prod_{i=0}^{n} q_{ij}$ *for* ${}^\forall j$.

Remark
When $d_i = 1$, then the thm is obtained by Kanazawa in 2014.

Ideas of the proof

1. Proving $\mathrm{qgr}(A)$ is sm.
   $\rightarrow$ We need the notion of quasi-Veronese algebras introduced by Mori.
2. Calculating $\mathcal{S}_{\mathrm{qgr}(A)}$
   $\rightarrow$ We use the theory of NC local cohomology and Serre functs by Yekutieli, Van den bergh and ....

Comparison & examples

We consider a CY 2-sch and choose $(d_0, d_1, d_2, d_3) = (1, 1, 2, 2)$ and

$$(q_{ij}) := \begin{pmatrix} 1 & 1 & 1 & \omega^2 \\ 1 & 1 & \omega^2 & 1 \\ 1 & \omega & 1 & 1 \\ \omega & 1 & 1 & 1 \end{pmatrix}, \quad \omega := \frac{-1+\sqrt{3}i}{2}.$$

Then, $\mathrm{Proj}_{\mathrm{nc}}(A)$ is NOT isomorphic to both comm CY and NC CYs by Kanazawa.

Result 2

Theorem (M)

$X := \mathrm{Proj}_{\mathrm{nc}}(\Delta(S \otimes T/(f_1, f_2)))$.

(i)

- $S = k[x_0, \cdots, x_n]_{(q_{ij})}$.
- $T = k[y_0, \cdots, y_m]_{(q'_{ij})}$.
- $f_1 = \Sigma x_i^{n+1}, f_2 = \Sigma y_j^{m+1}$.

Suppose

1. $q_{ii} = q_{ij} q_{ji} = q_{ij}^{n+1} = 1$
2. $q'_{ii} = q'_{ij} q'_{ji} = {q'}_{ij}^{m+1} = 1$

Then,
X *is* **CY $(n+m-2)$-sch**
iff $^{\exists} c, c' \in k^{\times}$ *s.t.*
$c = \Pi_{i=0}^{n} q_{ij}, c' = \Pi_{i=0}^{m} q'_{ij}$.

(ii)

- $S = k[x_0, \cdots, x_n]_{(q_{ij})}$.
- $T = k[y_0, \cdots, y_{n+1}]$.
- $f_1 = \Sigma x_i^{n+1} y_i, f_2 = \Sigma y_j^{n+1}$.

Suppose

$$q_{ii} = q_{ij} q_{ji} = q_{ij}^{n+1} = 1$$

Then,
X *is* **CY $(2n-1)$-sch**
iff $^{\exists} c \in k^{\times}$ *s.t.* $c = \Pi_{i=0}^{n} q_{ij}$.

※$\Delta(R) := \oplus_i R_{ii}$.

Ideas of the proof

$C := S \otimes T/(f_1, f_2)$.

We use the equivalnece $\mathrm{qbigr}(C) \simeq \mathrm{qgr}(\Delta(C))$ by Rompay.

Then,

1. Prove $\mathrm{qbigr}(C)$ is sm.
   $\rightarrow$ We can use the same idea in Result 1.
2. Calculating $\mathcal{S}_{\mathrm{qbigr}(C)}$
   $\rightarrow$ We need to construct some theory of NC local cohomology and Serre functors of $\mathbb{Z}^2$-gr algs.

Thank you for listening !