Bondal-Orlov's reconstruction theorem in noncommutative projective geometry

Yuki Mizuno

Waseda University

January, 2026

Plan of this talk

1. Introduction: reconstruction problems & noncommutative (NC) projective geometry
2. Main theorem and necessary notions
3. An application to the study of Artin-Schelter (AS) regular algebras
4. NC projective Calabi–Yau (CY) geometry: a related question and recent progress

In this talk, we work over a field k.

Introduction: reconstruction problems

Question

When can we reconstruct a scheme from
its (derived) category of coherent sheaves ?

Theorem (Gabriel '62)

X, Y : noetherian schs.
Then,

$$\mathrm{Coh}(X) \simeq \mathrm{Coh}(Y) \Rightarrow X \simeq Y.$$

Theorem (Bondal-Orlov '01)

X, Y : smooth proj vars over k.
Assume that the canonical bundles ω_X, ω_Y of X, Y are (anti-)ample.
Then,

$$D^b(\mathrm{Coh}(X)) \simeq D^b(\mathrm{Coh}(Y)) \Rightarrow X \simeq Y.$$

Rmk It suffices to assume that either ω_X or ω_Y is (anti-)ample.

Introduction : motivation for NC proj geometry

- $R = \bigoplus_{i \in \mathbb{N}} R_i$: commutative fin gen $\mathbb{N}$-gr k-alg w/ $R_0 = k$.
- $\mathrm{gr}(R)$: cat of fin gen $\mathbb{Z}$-gr R-mods
- $\mathrm{tor}(R)$: full subcat of $\mathrm{gr}(R)$ consisting of torsion mods

Definition

We define the quotient category $\mathrm{qgr}(R) := \mathrm{gr}(R)/\mathrm{tor}(R)$ as follows:

▷ $\mathrm{obj}(\mathrm{qgr}(R)) = \mathrm{obj}(\mathrm{gr}(R))$,

▷ $\mathrm{Hom}_{\mathrm{qgr}(R)}(\pi_R(M), \pi_R(N)) = \varinjlim_n \mathrm{Hom}_{\mathrm{gr}(R)}(M_{\geq n}, N_{\geq n})$,

where $\pi_R : \mathrm{gr}(R) \to \mathrm{qgr}(R)$: the projection functor.

Rmk

$\pi_R(M) \simeq \pi_R(N) \iff M_{\geq n} \simeq N_{\geq n}$ for $n \gg 0$.

Introduction : motivation for NC proj geometry

- $R = \bigoplus_{i \in \mathbb{N}} R_i$: commutative fin gen $\mathbb{N}$-gr k-alg w/$R_0 = k$.
- $\mathrm{qgr}(R) = \mathrm{gr}(R)/\mathrm{tor}(R)$: quotient cat.

Theorem (Serre '55)

Assume that R is generated by R_1 as a k-alg.
Then,

$$\mathrm{Coh}(\mathrm{Proj}(R)) \simeq \mathrm{qgr}(R).$$

By [Serre '55] + [Gabriel '62], we can think that

qgr(R) is essential in projective algebraic geometry !

Introduction: NC proj geometry

- $A = \bigoplus_{i \in \mathbb{N}} A_i$: locally finite (i.e. $\dim_k A_i < \infty$ for $^\forall i$)
  noeth $\mathbb{N}$-gr k-alg.
- $\mathrm{gr}(A)$: cat of fin gen $\mathbb{Z}$-gr right A-mods.
- $\mathrm{qgr}(A) = \mathrm{gr}(A)/\operatorname{tor}(A)$: quotient cat.

Definition (Artin-Zhang '94)

We call $\mathrm{qgr}(A)$ the NC projective scheme associated with A.
We often write $\mathcal{O}_A$ for $\pi_A(A) \in \mathrm{qgr}(A)$.

Theorem (M, rough statement)

Under appropriate conditions,
Bondal-Orlov's reconstruction theorem holds for NC proj schs.

Key notions

Canonical bundles, their (anti-)ampleness and dualizing complexes in NC world.

Canonical bimodules in the theory of abelian categories

$\mathcal{C}$: k-linear abelian cat, $F : \mathcal{C} \to \mathcal{C}$: autoequiv.

Definition (Mori-Ueyama '21)

F is a canonical bimodule on $\mathcal{C}$ if ${}^{\exists}n \in \mathbb{Z}$ s.t.

$$F[n] : D^b(\mathcal{C}) \to D^b(\mathcal{C})$$

is a Serre functor, i.e. $\mathrm{Hom}_{D^b(\mathcal{C})}(M, N) \simeq \mathrm{Hom}_{D^b(\mathcal{C})}(N, F[n](M))^\vee$.

E.g.

ω_X : canonical sheaf of a proj var X.

1. X : smooth $\Rightarrow - \otimes_{\mathcal{O}_X} \omega_X$: can bimod on $\mathrm{Coh}(X)$ & $n = \dim(X)$.
2. X : Calabi-Yau (i.e. smooth & $\omega_X \simeq \mathcal{O}_X$)
   $\Leftrightarrow \mathrm{id}_{\mathrm{Coh}(X)}$ is a can bimod of $\mathrm{Coh}(X)$.

Ampleness in the theory of abelian categories

$\mathcal{O}$: object in $\mathcal{C}$, $F : \mathcal{C} \to \mathcal{C}$: autoequiv.

Definition (Artin-Zhang '94)

A pair $(\mathcal{O}, F)$ is ample if

1. $^{\forall} M \in \mathcal{C}$, $^{\exists}$ epimor $\varphi : \bigoplus_{i=1}^{r} F^{-\ell_i}(\mathcal{O}) \twoheadrightarrow M \quad (\ell_1, \cdots, \ell_r \in \mathbb{N})$.
2. $^{\forall}$ epimor $f : M \to N$, $^{\forall} m \gg 0$,

$$\begin{array}{ccc} & & F^{-m}(\mathcal{O}) \\ & {\scriptstyle \exists g} \swarrow & \downarrow {\scriptstyle \forall h} \\ M & \xrightarrow{\ f\ } & N \end{array} \quad : \text{ commutative}$$

A pair $(\mathcal{O}, F)$ is anti-ample if $(\mathcal{O}, F^{-1})$ is ample.

E.g.

L : an invertible sheaf on a proj var X.

- L is ample on $X \Leftrightarrow (\mathcal{O}_X, - \otimes_{\mathcal{O}_X} L)$ is ample.

Dualizing complexes of NC graded algebras

A : locally fin noeth $\mathbb{N}$-gr k-alg

Definition (Yekutieli '92)

A dualizing complex (dc) of A is a cpx $R \in D^b(\mathrm{Gr}(A^{en}))$ s.t.

1. R has fin inj dim & fin gen cohomology over A & A^{op},
2. The functor

$$\mathbf{R}\operatorname{Hom}_A(-, R) : D^b(\mathrm{gr}(A))^{\mathrm{op}} \to D^b(\mathrm{gr}(A^{\mathrm{op}}))$$

is an equiv with inverse $\mathbf{R}\operatorname{Hom}_{A^{\mathrm{op}}}(-, R)$.

Moreover, R is balanced if $\mathbf{R}\Gamma_{\mathfrak{m}_A}(R) \simeq \mathbf{R}\Gamma_{\mathfrak{m}_{A^{\mathrm{op}}}}(R) \simeq A'$ (graded k-dual).

Rmk

A has a balanced dc & $\mathrm{qgr}(A)$ has a can bimod

$\Rightarrow \pi_A^*(- \otimes_A H^{-(d+1)}(R))$: can bimod of $\mathrm{qgr}(A)$,

where $d = \mathrm{gl.dim}(\mathrm{qgr}(A)) = \sup\{i \in \mathbb{Z} \mid \mathrm{Ext}^i_{\mathrm{qgr}(A)}(-, -) \neq 0\}$.

Main theorem

A, B : locally fin noeth $\mathbb{N}$-gr k-algs w/ balanced dc R_A, R_B.

Theorem (M)

Assume that $\mathrm{qgr}(A), \mathrm{qgr}(B)$ have canonical bimodules K_A, K_B.
If $(\mathcal{O}_A, K_A), (\mathcal{O}_B, K_B)$ are (anti-)ample, then

$$D^b(\mathrm{qgr}(A)) \simeq D^b(\mathrm{qgr}(B)) \Rightarrow \mathrm{qgr}(A) \simeq \mathrm{qgr}(B).$$

Rmk

- Main theorem + [Serre '55] + [Gabriel '62]
  $\Rightarrow$ the original Bondal-Orlov's thm.
- The proof of the main thm is different from the original one.

  Reason: It is difficult to deal with point-like objects in NC geometry.

  Key lemma: Any equiv $F : D^b(\mathrm{qgr}(A)) \to D^b(\mathrm{qgr}(B))$ is of FM type.
  $\rightsquigarrow$ we can compare the canonical algebras C_A, C_B of A, B.

AS regular algebras

A : connected (i.e. $A_0 = k$) noeth $\mathbb{N}$-gr k-alg.
$k = A/A_{>0}$ is regarded as an A-mod.

Definition (Artin-Schelter '87)

A is Artin-Schelter (AS) regular if

1. $d := \mathrm{gl.dim}(A) < \infty$,
2. A is Gorenstein, i.e.

$$\mathrm{Ext}^i_A(k, A) \simeq \begin{cases} 0 & (i \neq d) \\ k(\ell) & (i = d) \end{cases} \quad \text{in } \mathrm{gr}(A) \ \& \ \mathrm{gr}(A^{\mathrm{op}})$$

for some $\ell \in \mathbb{Z}$, called the Gorenstein parameter of A.

Rmk

- A : commutative AS reg alg $\Leftrightarrow$ A : polynomial ring.

Examples of AS regular algebras

Example

- 1-dim AS reg alg $\simeq k[t]$.
- 2-dim AS reg alg $\simeq \begin{cases} k\langle x,y\rangle/(xy-qyx) & (q\in k^*) \quad \text{or} \\ k\langle x,y\rangle/(xy-yx-y^m) & (m\in\mathbb{N}). \end{cases}$
- $k\langle x_1,\cdots,x_n\rangle/(x_ix_j-q_{ij}x_jx_i \mid 1\le i,j\le n)$, $(q_{ij}\in k^*, q_{ii}=q_{ij}q_{ji}=1)$.

Example (Sklyanin '83)

$[a:b:c]\in\mathbb{P}^2_k\setminus\{\text{finite pts}\}$.

$$S_{a,b,c} := k\langle x,y,z\rangle/(f_1,f_2,f_3),$$

$$f_1 = ayz+bzy+cx^2,$$

$$f_2 = azx+bxz+cy^2,$$

$$f_3 = axy+byx+cz^2.$$

There are much more (higher-dimensional) examples such as Feigin-Odesskii's elliptic algebras.

An application of main theorem

Corollary (M)

Let A, B be AS regular algebras.
Then,

$$D^b(\mathrm{qgr}(A)) \simeq D^b(\mathrm{qgr}(B)) \Rightarrow \mathrm{qgr}(A) \simeq \mathrm{qgr}(B).$$

Rmk

- To prove the corollary,
  we need to consider non-connected version of AS regular algs !

  Reason: The anti-ampleness of the can bimod is NOT obvious.

  Key notion: ℓ-th quasi-Veronese algebra $A^{[\ell]}$ of A

$$A^{[\ell]} := \bigoplus_{i \in \mathbb{N}} \begin{pmatrix} A_{\ell i} & A_{\ell i+1} & \cdots & A_{\ell i+\ell-1} \\ A_{\ell i-1} & A_{\ell i} & \cdots & A_{\ell i+\ell-2} \\ \vdots & \vdots & \ddots & \vdots \\ A_{\ell i-\ell+1} & A_{\ell i-\ell+2} & \cdots & A_{\ell i} \end{pmatrix}.$$

$\rightsquigarrow$ We have the equiv $\mathrm{qgr}(A) \simeq \mathrm{qgr}(A^{[\ell]})$.
It's easy to show the can bimod of $\mathrm{qgr}(A^{[\ell]})$ is anti-ample.

A question related to NC CY manifolds

Definition

$\mathbf{q} = (q_{ij})_{0\leq i,j\leq n}$: matrix over k^* with $q_{ii} = q_{ij}q_{ji} = 1$ for $^\forall i, j$.
$\mathbf{d} = (d_0, \cdots, d_n) \in \mathbb{N}^{n+1}$ satisfying $d_i \mid d_0 + \cdots + d_n$ $(=: d)$ for $^\forall i$.

$$k[x_0, \cdots, x_n]_{\mathbf{q},\mathbf{d}} := k\langle x_0, \cdots, x_n\rangle/(x_i x_j - q_{ij} x_j x_i \mid 0 \leq i, j \leq n),$$

with $\deg x_i = d_i$ for $^\forall i$.

Theorem (Kanazawa '14, M '24)

Let $A_{\mathbf{q},\mathbf{d}} := k[x_0, \cdots, x_n]_{\mathbf{q},\mathbf{d}}/(x_0^{d/d_0} + \cdots + x_n^{d/d_n})$.
Assume

1. $q_{ij}^{d/d_i} = q_{ij}^{d/d_j} = 1$ for $^\forall i, j$.
2. $^\exists c \in k^*$ s.t. $c^{d_j} = \prod_{i=0}^{n} q_{ij}$ for $^\forall j$.

Then, $\mathrm{qgr}(A_{\mathbf{q},\mathbf{d}})$ is CY, i.e. $\mathrm{qgr}(A_{\mathbf{q},\mathbf{d}})$ has a trivial canonical bimod.

A question related to NC CY manifolds

- $k[x_0, \cdots, x_n]_{\mathbf{q},\mathbf{d}} := k\langle x_0, \cdots, x_n\rangle/(x_i x_j - q_{ij} x_j x_i \mid 0 \le i, j \le n)$,
  with $\deg x_i = d_i$ for $^\forall i$.

Theorem (Kanazawa '14, M '24)

Let $A_{\mathbf{q},\mathbf{d}} := k[x_0, \cdots, x_n]_{\mathbf{q},\mathbf{d}}/(x_0^{d/d_0} + \cdots + x_n^{d/d_n})$.
Assume

1. $q_{ij}^{d/d_i} = q_{ij}^{d/d_j} = 1$ for $^\forall i, j$.
2. $^\exists c \in k^*$ s.t. $c^{d_j} = \prod_{i=0}^n q_{ij}$ for $^\forall j$.

Then, $\mathrm{qgr}(A_{\mathbf{q},\mathbf{d}})$ is CY, i.e. $\mathrm{qgr}(A_{\mathbf{q},\mathbf{d}})$ has a trivial canonical bimod.

Question 1

$A_{\mathbf{q}_1,\mathbf{d}_1}, A_{\mathbf{q}_2,\mathbf{d}_2}$: $\mathbb{N}$-gr algs which satisfy the assumptions of the above thm.

$$D^b(\mathrm{qgr}(A_{\mathbf{q}_1,\mathbf{d}_1})) \simeq D^b(\mathrm{qgr}(A_{\mathbf{q}_2,\mathbf{d}_2})) \Rightarrow \mathrm{qgr}(A_{\mathbf{q}_1,\mathbf{d}_1}) \simeq \mathrm{qgr}(A_{\mathbf{q}_2,\mathbf{d}_2}) \ ?$$

Updates of study on geometry of NC CY manifolds

We focus on the 2-dim case of the above construction.

Theorem (M j.w.w P. Belmans & O. van Garderen)

$A_{\mathbf{q},\mathbf{d}}$: as in the thm w/ $\mathrm{gl.dim}(\mathrm{qgr}(A_{\mathbf{q},\mathbf{d}})) = 2$.

Then, $\mathrm{qgr}(A_{\mathbf{q},\mathbf{d}})$ is a twisted NCCR (in the sense of M. Van den Bergh) over a singular K3 surface X only with ADE singularities.

E.g. When $\mathbf{q} = (q_{ij})$ and $\mathbf{d}$ are given by

$$q_{ij} = \begin{cases} 1 & (i = j) \\ -1 & (i \neq j) \end{cases}, \quad \mathbf{d} = (1, 1, 1, 1),$$

then, $X \cong \mathrm{Proj}(k[x_0, x_1, x_2, x_3, y]/(x_0^2 + x_1^2 + x_2^2 + x_3^2, x_0x_1x_2x_3 - y^2))$.
In addition, $^{\exists}\mathcal{A}$: a coh sheaf of algebras on X s.t.

$$\mathrm{qgr}(A_{\mathbf{q},\mathbf{d}}) \simeq \mathrm{Coh}(X, \mathcal{A})$$

and $\mathcal{A}$ is Azumaya except for the singular points of X.

Updates of study on geometry of NC CY manifolds

Corollary (M j.w.w P. Belmans & O. van Garderen)

$A_{\mathbf{q},\mathbf{d}}$: as in the thm w/ $\mathrm{gldim}(\mathrm{qgr}(A_{\mathbf{q},\mathbf{d}})) = 2$.
X_{can} : the canonical stack of X.

1. There is an Azumaya algebra $\mathcal{A}_{\mathrm{can}}$ over X_{can} s.t.

$$\mathrm{qgr}(A_{\mathbf{q},\mathbf{d}}) \simeq \mathrm{Coh}(X_{\mathrm{can}}, \mathcal{A}_{\mathrm{can}}).$$

2. We have the isomorphism of Hochschild homology:

$$\mathrm{HH}_\bullet(D^b(\mathrm{qgr}(A_{\mathbf{q},\mathbf{d}}))) \simeq \mathrm{HH}_\bullet(\text{K3 surfaces}).$$

Conjecture (Bondal-Orlov '02, Van den Bergh '04)

Any CRs and (untwisted) NCCRs are derived equivalent.

Question 2

Are $\mathrm{qgr}(A_{\mathbf{q},\mathbf{d}})$ derived equivalent to a twisted K3 surface ?

Thank you for your attention.

Theorem (M)

A, B : locally fin noeth $\mathbb{N}$-gr k-algs w/ balanced dc.
Assume that $\mathrm{qgr}(A), \mathrm{qgr}(B)$ have can bimods K_A, K_B.

If $(\mathcal{O}_A, K_A), (\mathcal{O}_B, K_B)$ are (anti-)ample, then

$$D^b(\mathrm{qgr}(A)) \simeq D^b(\mathrm{qgr}(B)) \Rightarrow \mathrm{qgr}(A) \simeq \mathrm{qgr}(B).$$

Corollary (M)

A, B : noeth AS regular algs.
Then,

$$D^b(\mathrm{qgr}(A)) \simeq D^b(\mathrm{qgr}(B)) \Rightarrow \mathrm{qgr}(A) \simeq \mathrm{qgr}(B).$$