Bondal-Orlov's reconstruction theorem in noncommutative projective geometry (arXiv:2411.07813)

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Introduction: Reconstruction Problems

k : alg clo field.

Question

When can we renconstruct a scheme from
its (derived) category of coherent sheaves ?

Theorem (Gabriel '62)

X, Y : noeth schs.
Then,

$$\mathrm{Coh}(X) \simeq \mathrm{Coh}(Y) \Rightarrow X \simeq Y.$$

Theorem (Bondal-Orlov '01)

X, Y : sm proj vars over k.
Assume that the canonical bundles of X, Y are (anti-)ample.
Then,

$$D^b(\mathrm{Coh}(X)) \simeq D^b(\mathrm{Coh}(Y)) \Rightarrow X \simeq Y.$$

Introduction: Motivation of NC Proj Geometry

- $R = \bigoplus_{i \in \mathbb{N}} R_i$: commutative fin gen $\mathbb{N}$-gr k-alg.
- $\mathrm{gr}(R)$: cat of fin gen $\mathbb{Z}$-gr R-mods.
- $\mathrm{qgr}(R)$: quotient cat of $\mathrm{gr}(R)$ by $\mathrm{tor}(R)$.
  - $\mathrm{obj}(\mathrm{qgr}(R)) = \mathrm{obj}(\mathrm{gr}(R))$.
  - $\mathrm{Hom}_{\mathrm{qgr}(R)}(\pi_R(M), \pi_R(N)) = \varinjlim_n \mathrm{Hom}_{\mathrm{gr}(R)}(M_{\geq n}, N_{\geq n})$,
    where $\pi_R : \mathrm{gr}(R) \to \mathrm{qgr}(R)$ is the projection.

Theorem (Serre '55)

Assume that R is generated by R_1 as an R_0-alg.
Then,

$$\mathrm{Coh}(\mathrm{Proj}(R)) \simeq \mathrm{qgr}(R).$$

qgr(*R*) is essential in projective algebraic geometry !

Introduction: NC Proj Schemes & Main Thm (Rough)

$A := \bigoplus_{i \in \mathbb{N}} A_i$: locally finite (i.e. $\dim_k A_i < \infty$) noeth $\mathbb{N}$-gr k-alg.

Definition (Artin-Zhang '94)

qgr(A) is the noncommutative projective (NC) scheme associated with A.

Theorem (M, rough version)

Under appropriate conditions,
Bondal-Orlov's reconstruction theorem holds for NC proj schs.

Key notions:

- canonical bundles of NC proj schs,
- (anti-)ampleness of canonical bundles of NC proj schs,
- dualizing complexes of NC graded algs.

Canonical Bimodules on Abelian Categories

$\mathcal{C}$: k-linear abelian cat, $F : \mathcal{C} \to \mathcal{C}$: autoequiv.

Definition (Mori-Ueyama '21)

F is a canonical bimodule on $\mathcal{C}$ if $^{\exists} n \in \mathbb{Z}$ s.t.

$$F[n] : D^b(\mathcal{C}) \to D^b(\mathcal{C})$$

is a Serre functor, i.e. $\mathrm{Hom}_{D^b(\mathcal{C})}(M, N) \simeq \mathrm{Hom}_{D^b(\mathcal{C})}(N, F[n](M))^\vee$.

E.g.

ω_X : can bdl of a sm proj var X.
Then, $- \otimes_{\mathcal{O}_X} \omega_X$ is a can bimod on $\mathrm{Coh}(X)$ and $n = \dim(X)$.

Remark

- A proj var X is sm $\Leftrightarrow$ $\mathrm{Coh}(X)$ has a can bimod.

Ampleness in the Theory of Abelian Categories

$\mathcal{O}$: obj in $\mathcal{C}$, $F : \mathcal{C} \to \mathcal{C}$: autoequiv.

Definition (Artin-Zhang '94)

A pair $(\mathcal{O}, F)$ is ample if

1. ${}^{\forall}M \in \mathcal{C}$, ${}^{\exists}$ an epimor φ

$$\varphi : \bigoplus_{i=1}^{r} F^{-\ell_i}(\mathcal{O}) \to M \quad (\ell_1, \cdots, \ell_r \in \mathbb{N}).$$

2. ${}^{\forall}$ epimor $f : M \to N$, ${}^{\exists}m_0 \in \mathbb{N}$ s.t. the natural mor

$$\mathrm{Hom}_{\mathcal{C}}(F^{-m}(\mathcal{O}), M) \to \mathrm{Hom}_{\mathcal{C}}(F^{-m}(\mathcal{O}), N).$$

   is surj for ${}^{\forall}m \geq m_0$.

A pair $(\mathcal{O}, F)$ is anti-ample if $(\mathcal{O}, F^{-1})$ is ample.

E.g.

L : a line bdl on a sm proj var X.

Then, L is ample on $X \Leftrightarrow (\mathcal{O}_X, - \otimes_{\mathcal{O}_X} L)$ is ample.

Dualizing Complexes of NC Graded Algebras

Definition (Yekutieli '92)

A dualizing complex (dc) of A is a cpx $R \in D^b(\mathrm{Gr}(A^{en}))$ s.t.

1. R has fin inj dim & fin gen cohomology over A & A^{op},
2. The functor

$$\mathbf{R}\operatorname{Hom}_A(-, R) : D^b(\mathrm{gr}(A)) \to D^b(\mathrm{gr}(A^{\mathrm{op}}))$$

is an equiv with inverse $\mathbf{R}\operatorname{Hom}_{A^{\mathrm{op}}}(-, R)$.

Moreover, R is balanced if $\mathbf{R}\Gamma_{\mathfrak{m}_A}(R) \simeq \mathbf{R}\Gamma_{\mathfrak{m}_{A^{\mathrm{op}}}}(R) \simeq A'$ (graded k-dual).

Remark

- A has a balanced dc & $\mathrm{qgr}(A)$ has a can bimod
  $\Rightarrow \pi_A(- \otimes_A H^{-(n+1)}(R))$: can bimod of $\mathrm{qgr}(A)$.

Main Theorem

A, B : loc fin noeth $\mathbb{N}$-gr k-algs w/ balanced dc R_A, R_B.

Theorem (M)

Assume that $\mathrm{qgr}(A), \mathrm{qgr}(B)$ have canonical bimodules K_A, K_B.
If $(\pi_A(A), K_A), (\pi_B(B), K_B)$ are (anti-)ample, then

$$\boldsymbol{D^b(\mathrm{qgr}(A)) \simeq D^b(\mathrm{qgr}(B)) \Rightarrow \mathrm{qgr}(A) \simeq \mathrm{qgr}(B)}.$$

Remark

- Main theorem $\Rightarrow$ the original Bondal-Orlov reconstruction.
- In the prf, showing the following two claims are crucial.
  1. Equivs between $D^b(\mathrm{qgr}(A))$ & $D^b(\mathrm{qgr}(B))$ are of Fourier-Mukai type.
  2. The canonical alg of A

$$C_A := \bigoplus_{m \in \mathbb{N}} H^0(\mathrm{qgr}(A), K_A^m(\pi_A(A)))$$

is isomorphic to C_B.

AS Regular Algebras

A : connected (i.e. $A_0 = k$) fin gen $\mathbb{N}$-gr k-alg.
$k = A/A_{>0}$ is regarded as an A-mod.

Definition (Artin-Schelter '87)

A is Artin-Schelter (AS) regular of dim d & Gorenstein para ℓ if

1. $\mathrm{gl.dim}(A) = d < \infty$,
2. $\{\dim_k A_i\}_{i \in \mathbb{N}}$ has poly growth.
3. A is Gorenstein, i.e. $\mathrm{Ext}^i_A(k, A) \simeq \begin{cases} 0 & (i \neq d) \\ k(\ell) & (i = d) \end{cases}$.

E.g.

- commutative AS reg alg $\simeq$ polynomial alg.
- 1-dim AS reg alg $\simeq k[t]$.
- 2-dim AS reg alg $\simeq k\langle x, y\rangle/(xy - qyx)$ or $k\langle x, y\rangle/(xy - yx - y^m)$.
  $(q \in k^\times, m \in \mathbb{N})$
- $k\langle x_1, \cdots, x_n\rangle/(x_i x_j - q_{ij} x_j x_i \mid 1 \leq i, j \leq n)$, $(q_{ij} \in k^*, q_{ii} = q_{ij} q_{ji} = 1)$.

More Examples of AS Regular Algebras

$S_{a,b,c}$ & $Q_{n,k}(E,\eta)$ are AS regular for gen a, b, c and η.

Example (Sklyanin '83)

Let $a, b, c \in k$.

$$S_{a,b,c} := k\langle x, y, z\rangle/(f_1, f_2, f_3),$$

$$f_1 = ayz + bzy + cx^2,\ f_2 = azx + bxz + cy^2,\ f_3 = axy + bxy + cz^2.$$

Example (Feigin-Odesskii '89, Chirvasitu-Kanda-Smith '23)

E : elliptic curve, $\eta \in E$: closed pt.

$1 \leq k < n$: coprime integers.

$$Q_{n,k}(E,\eta) := k\langle x_1, \cdots, x_n\rangle/(g_{i,j} \mid i, j \in \mathbb{Z}/n\mathbb{Z}),$$

$$g_{i,j} := \sum_{r \in \mathbb{Z}/n\mathbb{Z}} \frac{\theta_{j-i+(k-1)r}(0)}{\theta_{j-i-r}(-\eta)\theta_{kr}(\eta)} x_{j-r}x_{i+r},$$

where θ_α ($\alpha \in \mathbb{Z}/n\mathbb{Z}$) are certain theta functions.

An Application of Main Thoeorem

Corollary (M)

Let A, B be noetherian AS regular algebras.
Then,

$$D^b(\mathrm{qgr}(A)) \simeq D^b(\mathrm{qgr}(B)) \Rightarrow \mathrm{qgr}(A) \simeq \mathrm{qgr}(B).$$

Remark

- The cor holds for locally fin ver of AS regular algs.
- **Even when proving the connected case,**
  we need to consider locally fin ver of AS regular algs !

The Proof of Corollary

ℓ_A, ℓ_B : Gorenstein paras of A, B.

1. Taking a quasi-Veronese algebra $A^{[\ell_A]}$.

$$A^{[\ell_A]} := \bigoplus_{i \in \mathbb{N}} A_i^{[\ell_A]} := \bigoplus_{i \in \mathbb{N}} \begin{pmatrix} A_{\ell_A i} & A_{\ell_A i+1} & \cdots & A_{\ell_A i+\ell_A-1} \\ A_{\ell_A i-1} & A_{\ell_A i} & \cdots & A_{\ell_A i+\ell_A-2} \\ \vdots & \vdots & \ddots & \vdots \\ A_{\ell_A i-\ell_A+1} & A_{\ell_A i-\ell_A+2} & \cdots & A_{\ell_A i} \end{pmatrix}.$$

2. We have the equiv $\mathrm{qgr}(A) \simeq \mathrm{qgr}(A^{[\ell_1]})$.
3. $\ell_{A^{[\ell_A]}} = 1$ (Mori-Minamoto) $\rightsquigarrow$ $(\pi_{A^{[\ell_A]}}(A^{[\ell_A]}), K_{A^{[\ell_A]}})$: anti-ample.
4. We can apply the thm to $\mathrm{qgr}(A^{[\ell_A]})$ & $\mathrm{qgr}(B^{[\ell_B]})$.

Remark

- $A^{[\ell_A]}$ is a locally fin ver of AS regular algs.
- Anti-ampleness of K_A is NOT obvious without considering $A^{[\ell_A]}$.

Theorem (M)

A, B : loc fin noeth $\mathbb{N}$-gr k-algs w/ balanced dc.
Assume that $\mathrm{qgr}(A), \mathrm{qgr}(B)$ have can bimods K_A, K_B.

If $(\pi_A(A), K_A), (\pi_B(B), K_B)$ are (anti-)ample, then

$$\boldsymbol{D^b(\mathrm{qgr}(A)) \simeq D^b(\mathrm{qgr}(B)) \Rightarrow \mathrm{qgr}(A) \simeq \mathrm{qgr}(B)}.$$

Corollary (M)

A, B : noeth AS regular algs.
Then,

$$\boldsymbol{D^b(\mathrm{qgr}(A)) \simeq D^b(\mathrm{qgr}(B)) \Rightarrow \mathrm{qgr}(A) \simeq \mathrm{qgr}(B)}.$$

Thank you for your attention.

The Core Idea of the Proof of Main Theorem

$F : D^b(\mathrm{qgr}(A)) \to D^b(\mathrm{qgr}(B))$: equiv.
Assume that $(\pi_A(A), K_A), (\pi_B(B), K_B)$ are ample.

1. F is of Fourier-Mukai type, i.e. ${}^{\exists}\mathcal{F} \in D(\mathrm{qbigr}(A^{\mathrm{op}} \otimes_k B))$ s.t.

$$F(-) \simeq \Phi_{\mathcal{F}}(-) := \pi_B(\mathbf{R}\omega_A(-) \otimes^{\mathbb{L}}_{\mathcal{A}} \mathbf{R}\omega_{A^{\mathrm{op}} \otimes_k B}(\mathcal{F})).$$

2. We have an iso of the graded k-algs

$$\begin{aligned} C_A &:= \bigoplus_{m \in \mathbb{N}} H^0(\mathrm{qgr}(A), K_A^m(\pi_A(A))) \\ &\simeq \bigoplus_{m \in \mathbb{N}} H^0(\mathrm{qgr}(B), K_B^m(\pi_B(B))) =: C_B. \end{aligned}$$

3. Finally, we have

$$\mathrm{qgr}(A) \simeq \mathrm{qgr}(C_A) \simeq \mathrm{qgr}(C_B) \simeq \mathrm{qgr}(B)$$

by the ampleness of K_A, K_B and Artin-Zhang's theorem.