CLASSIFYING THE IRREDUCIBLE COMPONENTS OF MODULI STACKS OF TORSION FREE SHEAVES ON K3 SURFACES AND AN APPLICATION TO BRILL-NOETHER THEORY

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ABSTRACT. In this article, we classify the irreducible components of moduli stacks of torsion free sheaves of rank 2 on K3 surfaces of Picard number 1. For ruled surfaces, the components of moduli stacks of torsion free sheaves were classified by Walter ([Wal95]). Moreover, by virtue of our result, we classify the irreducible components of Brill-Noether loci of Hilbert schemes of points on K3 surfaces.

1. INTRODUCTION

Moduli spaces of sheaves is one of the most central areas of algebraic geometry. By considering them, many interesting objects have been found. On K3 surfaces, moduli spaces of sheaves can have symplectic structures, which was first observed by Mukai ([Muk84b]). On the other hand, as is well-known, we can construct such moduli spaces by restricting objects to coherent sheaves satisfying stability. However, the moduli spaces do not parametrize unstable sheaves. In this point, stack is important and useful tool to construct moduli spaces which is difficult to construct in the framework of scheme.

Our original motivation of the present paper is studying symplecticity of moduli spaces of sheaves on K3 surfaces. Moreover, in [Muk84a] and [Yos99a] et al., it was shown that non-emptiness, irreducibility and other properties of moduli schemes depend essentially on Mukai vector. In [KY08] and [Yos03], properties of the moduli stacks of semistable sheaves on K3 surfaces are studied. Although we can study moduli spaces of unstable sheaves on K3 surfaces by using stack theory, detailed observations are less than studies of moduli schemes.

Various types of stratifications of stacks are studied by Gómez, Sols and Zamora [GSZ15] and Hoskins [Hos18] et al. However, it seems that irreducible decomposition of moduli stacks of sheaves is not treated in these papers. In the present article, we first classify the irreducible components of moduli stacks of torsion free sheaves of rank 2 on K3 surfaces of Picard number $\rho = 1$. Classifying the irreducible components of moduli stacks of torsion free sheaves on ruled surfaces is discussed in [Wal95]. However, we need new ideas to solve our problem because K3 surfaces have trivial canonical sheaves and may not be fibered surfaces. Important results and methods in this paper are studies of moduli stacks of semistable sheaves and filtered sheaves by Yoshioka ([KY11], [KY08], [Yos03], [Yos09]), the classical theory by Shatz ([Sha77]) and generalized Shatz's theory by Nitsure ([Nit11]). By using these theories, we obtain our first result. More precisely, we first take stratification of moduli stacks of torsion free sheaves by moduli stacks of semistable sheaves and ones of Harder-Narasimhan filtrations. After that, we analyze the strata and describe the irreducible components by using the above theory of Yoshioka.

If $\mathscr{M}^{\mathrm{tf}}(v)$ and $\mathscr{M}^{\mathrm{ss}}(v)$ denote respectively the moduli stacks of torsion free sheaves and semistable sheaves with Mukai vector v (in detail, see Definition 2.1), our first result is the following.

Theorem 1.1. *Let X be a K3 surface of $\rho(X) = 1$ over $\mathbb{C}$, let v_0 be a primitive Mukai vector and, let $v = ([v]_0, [v]_1, [v]_2) := mv_0$ $(m \in \mathbb{Z})$. We assume $[v]_0 = 2$. Then, we have the irreducible decomposition*

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of $\mathscr{M}^{\mathrm{tf}}(v)$ *as follows.*

$$\mathscr{M}^{\mathrm{tf}}(v) = \begin{cases} \overline{\mathscr{M}^{\mathrm{ss}}(v)} \cup \bigcup_{(v_1,v_2) \leq 1} \overline{\mathscr{M}^{HN}_{(v_1,v_2)}(v)} & \textit{if } \langle v_0, v_0 \rangle \geq -2 \\ \bigcup \overline{\mathscr{M}^{HN}_{(v_1,v_2)}(v)} & \textit{otherwise} \end{cases}$$

, where the stack $\mathscr{M}^{\mathrm{HN}}_{(v_1,v_2)}(v)$ *is defined as*

$$\mathscr{M}^{\mathrm{HN}}_{(v_1,v_2)}(v) := \left\{ E \in \mathscr{M}^{\mathrm{tf}}(v) \left| \begin{array}{c} \exists (0 \subset E_1 \subset E) : \textit{Harder-Narasimhan filtration} \\ \textit{such that } v(E_1) = v_1, v(E/E_1) = v_2 \end{array} \right. \right\}.$$

We call $\mathscr{M}^{\mathrm{HN}}_{(v_1,v_2)}(v)$ the moduli stack of Harder-Narasimhan filtrations with type (v_1, v_2). (in detail, see Definition 3.2)

Remark 1.2. Note that $\mathscr{M}^{\mathrm{ss}}(v) \neq \emptyset$ if and only if $\langle v_0, v_0 \rangle \geq -2$ ([Yos99a, Corollary 0.3]). And, we can compute the dimensions of $\mathscr{M}^{\mathrm{tf}}(v)$ at each point by using Theorem 1.1 and Lemma 3.8.

The second purpose of this paper is classifying the irreducible components of Brill-Noether loci of Hilbert schemes of points on K3 surfaces by using the first result. Originally, in [Wal95], components of Brill-Noether loci of Hilbert schemes of points on ruled surfaces were classified. In [Wal95], Castelnuovo-Mumford regularity and the Bertini's theorem were mainly used. However, we need more detailed analysis to achieve the application for K3 surfaces. Namely, we focus on the method of the proof of the Bertini theorem ([Bad12]) and more recent results about K3 surfaces ([KY08], [Yos99a]). Our second result is the following.

Theorem 1.3. *Let* X *be a K3 surface of* $\rho(X) = 1$ *over* $\mathbb{C}$, *let* $v := (2, nH, \frac{n^2}{2}H^2 - N + 2) = mv_0$ *(*v_0 : *primitive Mukai vector,* $m \in \mathbb{Z}$*) and let* nH *be an effective divisor on* X *(*$n \in \mathbb{Z}_{\geq 0}$, H : *the generator of* $\mathrm{Pic(X)}$*). We assume* $N \leq h^0(\mathscr{O}(nH))$. *Then, we classify the irreducible components of*

$$W_N^0(nH) = \{[Z] \in \mathrm{Hilb}^N(X) \mid h^1(\mathscr{I}_Z(nH)) \geq 1\}$$

into one of the following.

(α) : *for all* (v_1, v_2), *if* $\langle v_1, v_2 \rangle \leq 1$, $[v_1]_1, [v_2]_1 \neq 0$: *effective and* $-1 < [v_2]_2$, *there exists a unique irreducible component of* $W_N^0(nH)$ *such that, for a general member* Z, *the torsion free sheaf* E *fitting into the extension*

$$0 \rightarrow \mathscr{O}_X \rightarrow E \rightarrow \mathscr{I}_Z(nH) \rightarrow 0$$

is contained in $\mathscr{M}^{HN}_{(v_1,v_2)}(v)$.

(β) : *if* $\langle v_0, v_0 \rangle \geq -2$ *except for* $H^2 = 2$ *and* $v = (2, 3H, 5)$, *there exists a unique irreducible component of* $W_N^0(nH)$ *such that for a general member* Z, *the torsion free sheaf* E *fitting into the extension*

$$0 \rightarrow \mathscr{O}_X \rightarrow E \rightarrow \mathscr{I}_Z(nH) \rightarrow 0$$

is contained in $\mathscr{M}^{ss}(v)$.

Remark 1.4. If $N > h^0(\mathscr{O}(nH))$, then $W_N^0(nH) = \mathrm{Hilb}^N(X)$. And by using Theorem 1.3, we see not only whether $W_N^0(nH)$ is empty or not but also the dimensions and the number of the irreducible components of $W_N^0(nH)$.

2. PRELIMINARIES

In this paper, the word *a surface* means a two-dimensional algebraic variety over $\mathbb{C}$. The word *an algebraic stack* means an Artin stack over $\mathbb{C}$. In addition, the word *open (resp. closed, resp. locally closed) substack* means a strictly substack whose inclusion map is an open (resp. closed, resp. locally closed) immersion (in detail, see [LMB00] or [Sta]).

2.1. Mukai vectors.

Definition 2.1 (Mukai vectors [HL10]). Let X be a K3 surface and let E be a coherent sheaf on X.Then the Mukai vector $v(E)$ of E is $(\mathrm{rank}(E), c_1(E), \frac{c_1(E)^2}{2} - c_2(E) + \mathrm{rank}(E)) \in \mathbb{Z} \oplus \mathrm{Pic}(X) \oplus \mathbb{Z}$.

Definition 2.2 (Mukai paring [HL10]). Let X be a K3 surface and let $v := ([v]_0, [v]_1, [v]_2)$, $v' := ([v']_0, [v']_1, [v']_2) \in \mathbb{Z} \oplus \mathrm{Pic}(X) \oplus \mathbb{Z}$. Then, we define the Mukai pairing of v and v' to be $\langle v, v' \rangle :=$ $-[v]_0[v']_2 + [v]_1[v']_1 - [v]_2[v']_0 \in \mathbb{Z}$.

Definition 2.3 (([HL10])). For any $v \in \mathbb{Z} \oplus \mathrm{Pic}(X) \oplus \mathbb{Z}$, v is primitive if " $v' \in \mathbb{Z} \oplus \mathrm{Pic}(X) \oplus \mathbb{Z}$, $m \in \mathbb{Z}$, $v = mv' \Rightarrow m = 1$ or -1 "

2.2. Moduli stacks.

Definition 2.4 (Moduli stacks of torsion free sheaves). Let X be a K3 surface over $\mathbb{C}$, and let v $\in \mathbb{Z} \oplus \mathrm{NS}(X) \oplus \mathbb{Z}$. we define the moduli stack $\mathscr{M}^{\mathrm{tf}}(v)$ of torsion free sheaves with Mukai vector v on X to be the following category

(1) Objects: (S, E), where S : scheme over $\mathbb{C}$, E : quasi-coherent locally of finite presentation sheaves over $X \times_{\mathbb{C}} S(=: Z)$ and flat over S, and E_t : torsion-free sheaf over $Z_t = X_{k(t)}$ such that $v(E) = v$, $(\forall t \in S)$;

(2) Morphisms : morphisms from (S, E) to (S', E') are the pairings $(\varphi : S \to S', \alpha : \varphi^* E \to E')$ such that φ is an isomorphism.

Remark 2.5. $\mathscr{M}^{tf}(v)$ is an algebraic stack. And, we can define moduli stacks $\mathscr{M}(v)$ of coherent sheaves with Mukai vector v on X in the same way.

Definition 2.6 (Points of algebraic stacks [LMB00], [Sta]). Let $\mathscr{X}$ be an algebraic stack. Then,

$$|\mathscr{X}| := \coprod_{K/\mathbb{C}:\text{extension of fields}} \mathscr{X}(\mathrm{Spec}(K))/ \sim,$$

where if let $E \in \mathscr{X}(\mathrm{Spec}(K))$, let $E' \in \mathscr{X}(\mathrm{Spec}(K'))$ and let K, K' be extensions of $\mathbb{C}$, we write $E \sim E'$ if there exists a extension K'' of K, K' such that $E \mid_{X_{\mathrm{Spec}(K'')}} \simeq E' \mid_{X_{\mathrm{Spec}(K'')}}$.

Definition 2.7 (Topological spaces of algebraic stacks [LMB00], [Sta]). Let $\mathscr{X}$ be an algebraic stack. Then the set $\{U \subseteq |\mathscr{X}| \mid \exists \mathscr{U}:$ open substack of $\mathscr{X}$ such that $|\mathscr{U}| = U\}$ satisfies the axiom of open sets of $\mathscr{X}$. We think of $|\mathscr{X}|$ as a topological space by applying the definition.

Definition 2.8 (Relative dimensions [LMB00],[Sta]). Let $P : U \to \mathscr{X}$ be a morphism from a scheme, and we assume $u \in U$ maps to $x \in |\mathscr{X}|$. Then, we define $\dim_u(P)$ as follows. In the commutative diagram

$$\begin{array}{ccc} U \times_{\mathscr{X}} \mathrm{Spec}(k) & \longrightarrow & \mathrm{Spec}(k) \\ \downarrow & \square & \downarrow x \\ U & \xrightarrow{\quad P \quad} & \mathscr{X}, \end{array}$$

$$\dim_u(P) := \dim_x(U \times_{\mathscr{X}} \mathrm{Spec}(k)).$$

Definition 2.9 (Dimensions of algebraic stacks at points [LMB00], [Sta]). Let $\mathscr{X}$ be an algebraic stack, let $x \in \mathscr{X}(\mathrm{Spec}(K))$ where $K/\mathbb{C}$ is an extension and let $P : U \to \mathscr{X}$ be a smooth morphism from a scheme. We assume $u \in U$ maps to $x \in |\mathscr{X}|$. Then

$$\dim_x(\mathscr{X}) := \dim_u(U) - \dim_u(P).$$

Remark 2.10. If there is no confusion, we do not distinguish $\mathscr{X}$ with $|\mathscr{X}|$. And, Irreducible decomposition of $\mathscr{X}$ means irreducible decomposition of $|\mathscr{X}|$.

2.3. **Harder-Narasimhan filtrations and polygons.**

Theorem 2.11 (Harder-Narasimhan(HN) filtration [HL10])**.** *Let X be a projective surface over $\mathbb{C}$, let H be an ample divisor on X and let E be a torsion free sheaf on X. Then, for E and H, there exists a unique filtration*

$$0 = E_0 \subset E_1 \subset \cdots \subset E_{s-1} \subset E_s = E$$

such that E_i/E_{i-1} is μ-semistable for H $(i = 1, \cdots s)$ and $\mu(E_1/E_0) > \mu(E_2/E_1) > \cdots > \mu(E_{s-1}/E_{s-2}) > \mu(E_s/E_{s-1})$. It is called Harder-Narasimhan(HN) filtration of E for μ-stability.

In the same way, we have Harder-Narasimhan filtration of E for stability.

Definition 2.12 (Harder-Narasimhan polygon[Nit11],[Sha77])**.** Let X be a projective surface over $\mathbb{C}$, let H be an ample divisor on X and let E be a torsion free sheaf on X. We assume that E has the HN filtration for μ-stability

$$0 = E_0 \subset E_1 \subset \cdots \subset E_{s-1} \subset E_s = E.$$

Then, we define the Harder-Narasimhan(HN) polygon $\mathrm{HNP}(E)$ of E to be the polygon whose vertexes are $(0,0)$, $(\mathrm{rank}(E_1), \deg_H(E_1))$, $(\mathrm{rank}(E_1) + \mathrm{rank}(E_2), \deg_H(E_2))$, $\cdots$, $(\mathrm{rank}(E_1) + \cdots + \mathrm{rank}(E_{s-1})$, $\deg_H(E_{s-1}))$, $(\mathrm{rank}(E), \deg_H(E))$.

Remark 2.13. We can also define the HN polygon of E for stability. We use the notions of HN-polygon for both stability and μ-stability. (in detail, see [Nit11])

3. Irreducible decomposition of $\mathscr{M}^{\mathrm{tf}}(v)$

Notation 3.1. In this and next section, X always means a K3 surface of $\rho(X) = 1$ and H means the ample generator of $\mathrm{Pic}(X)$. We denote the open substack of semi stable sheaves and of μ-semi stable sheaves of $\mathscr{M}^{\mathrm{tf}}(v)$ by $\mathscr{M}^{\mathrm{ss}}(v)$ and $\mathscr{M}^{\mu\mathrm{ss}}(v)$. If $\overline{\{p\}} \ni p'$, then we write $p \rightsquigarrow p'$, where p, p' denote points of a topological space and say that p specializes p'.

Definition 3.2. we define $\mathscr{M}^{\mathrm{HN}}_{(v_1,v_2)}(v)$ to be a substack of $\mathscr{M}^{\mathrm{tf}}(v)$ whose objects and morphisms are defined as follows.

Objects: $E \in \mathscr{M}^{\mathrm{tf}}(v)$ such that E's HN-filtration is $0 \subset E_1 \subset E$ with $v(E_1) = (r_1, d_1 H, a_1)$, $v(E/E_1) = (r_2, d_2 H, a_2)$, where $v_i := (1, d_i H, a_i) \in \mathbb{Z} \oplus \mathrm{Pic}(X) \oplus \mathbb{Z}, (i = 1, 2)$;

Morphisms: $\alpha : E \to E'$: an isomorphism preserving their HN-filtrations.

Notation 3.3. Let v be an elment of $\mathbb{Z} \bigoplus \mathrm{Pic}(Z) \bigoplus \mathbb{Z}$. We define

$$Quot_X(F, v) := \{F \twoheadrightarrow E \mid E : \text{coherent on } X, v(E) = v\},$$
$$R^{N,m}(v) := \{\varphi : \mathscr{O}_X(-m)^{\oplus N} \twoheadrightarrow E \in Quot_X(\mathscr{O}_X(-m)^{\oplus N}, v) \mid H^0(\varphi(m)) : \text{isomorphism}\},$$
$$R^{N,m}_{\mathrm{tf}} := R^{N,m} \times_{\mathscr{M}(v)} \mathscr{M}^{\mathrm{tf}}(v),$$
$$R^{N,m}_{(v_1,v_2)} := R^{N,m} \times_{\mathscr{M}(v)} \mathscr{M}^{\mathrm{HN}}_{(v_1,v_2)}(v) \simeq R^{N,m} \times_{\mathscr{M}^{\mathrm{tf}}(v)} \mathscr{M}^{\mathrm{HN}}_{(v_1,v_2)}(v).$$

3.1. **Irreducibility of moduli stacks of sheaves and known results.** In this subsection, we refer to irreducibility of moduli stacks of HN-filtrations and known results needed to prove the our results.

Lemma 3.4 ([Yos03, Theorem 1.2])**.** *Let X be a K3 surface of Picard number 1. If $\langle v, v \rangle > 0$, then $\mathscr{M}^{\mathrm{ss}}(v)$ is an irreducible algebraic stack.* $\square$

Remark 3.5. When $\langle v, v \rangle \leq 0$ and $\mathscr{M}^{\mathrm{ss}}(v) \neq \emptyset$, the topological spaces of moduli stacks and moduli schemes are homeomorphic because the stacks are quotient stacks and all semistable sheaves are polystable. Therefore, the moduli stacks are irreducible.

Lemma 3.6 ([Yos09, Lemma 2.5])**.** *Let $\mathscr{M}^{HN}_{(v_1,v_2)}(v)$ be the moduli stack of torsion-free sheaves with Mukai vector v whose Harder-Narasimhan type is (v_1, v_2). Then*

(1) the morphism $\mathscr{M}^{HN}_{(v_1,v_2)}(v) \to \mathscr{M}(v)$ is an immersion;

(2) Let $E \in \mathscr{M}^{HN}_{(v_1,v_2)}(v)$, whose HN-filtration is $0 \to F_1 \to E \to F_2 \to 0$ and let $\mathscr{M}^{HN}_{(v_1,v_2)}(v) \to \mathscr{M}^{ss}(v_1) \times \mathscr{M}^{ss}(v_2)$ be a morphism which sends $[E] \mapsto ([F_1], [F_2])$. Then, all irreducible components of $\mathscr{M}^{HN}_{(v_1,v_2)}(v)$ are obtained as the pullback of an irreducible component of $\mathscr{M}^{ss}(v_1) \times \mathscr{M}^{ss}(v_2)$.

Corollary 3.7. $\mathscr{M}^{HN}_{(v_1,v_2)}(v)$ is an irreducible algebraic stack.

We explain facts about $\dim \mathscr{M}^{ss}(v)$ and $\dim \mathscr{M}^{HN}_{(v_1,v_2)}(v)$, which is necessary to prove a proposition later.

Lemma 3.8 ([KY11, Theorem 1.2] [KY08, Lemma 5.3], [MYY18, Lemma 5.3.2]). *Let X be K3 surface of Picard number 1. $v = lv_0$ with $v_0 :$ primitive$v$ and $l \in \mathbb{Z}$. Then,*

$$\dim \mathscr{M}^{ss}(v) = \begin{cases} \langle v, v \rangle + 1 & \langle v, v \rangle > 0 \\ \langle v, v \rangle + l & \langle v, v \rangle = 0 \\ \langle v, v \rangle + l^2 & \langle v_0, v_0 \rangle = -2 \end{cases}, \quad \dim \mathscr{M}^{HN}_{(v_1,v_2)}(v) = \langle v_1, v_1 \rangle + \langle v_2, v_2 \rangle + \langle v_1, v_2 \rangle + 2.$$

We also explain facts which is necessary to prove Theorem 1.1.

Lemma 3.9 (([KY11, Proposition 1.1])). *The dimensions of all irreducible components of $\mathscr{M}(v)$ is more than(or equal to) $\langle v, v \rangle + 1$.*

Lemma 3.10 ([EG17, Lemma 2.21]). *Let $\mathscr{X}$ be a pseudo-catenary, jacobson, and locally noetherian algebraic stack. If $|\mathscr{X}|$ is irreducible, then $\dim_x \mathscr{X}$ is constant for all $x \in |\mathscr{X}|$.*

Remark 3.11. (i) algebraic stacks which are locally of finite type satisfy the assumption of Lemma 3.10.

(ii) by Lemma 3.9 and Lemma 3.10, we get $\dim_x \mathscr{M}(v)^{tf} \geq \langle v, v \rangle + 1$ ($\forall x \in |\mathscr{M}(v)^{tf}|$).

3.2. **A criterion of the irreducible components of $\mathscr{M}^{tf}(v)$.** In this section, we classify the irreducible components of $\mathscr{M}^{tf}(v)$.

Lemma 3.12. *For the stacks $\mathscr{M}^{HN}_{(v_1,v_2)}(v)$ and $\mathscr{M}^{HN}_{(v'_1,v'_2)}(v)$, if $\dim \mathscr{M}^{HN}_{(v_1,v_2)}(v) \leq \dim \mathscr{M}^{HN}_{(v'_1,v'_2)}(v)$, then $\overline{\mathscr{M}^{HN}_{(v_1,v_2)}(v)} \nsubseteq \overline{\mathscr{M}^{HN}_{(v'_1,v'_2)}(v)}$.*

Proof. We assume that $\overline{\mathscr{M}^{HN}_{(v_1,v_2)}(v)} \subseteq \overline{\mathscr{M}^{HN}_{(v'_1,v'_2)}(v)}$ holds. If let p and p' be the generic points of $\overline{\mathscr{M}^{HN}_{(v_1,v_2)}(v)}$ and $\overline{\mathscr{M}^{HN}_{(v'_1,v'_2)}(v)}$ respectively, there exist $N, m \in \mathbb{Z}_{\geq 0}$ such that the morphism $R^{N,m}_{(v_1,v_2)} \to \overline{\mathscr{M}^{HN}_{(v_1,v_2)}(v)}$ is dominant, i.e., $R^{N,m}_{(v_1,v_2)} \ni \exists q \mapsto p \in \overline{\mathscr{M}^{HN}_{(v_1,v_2)}(v)}$. Note that $R^{N,m}_{(v_1,v_2)}$ is irreducible.

By the fact that $p' \rightsquigarrow p$ and [LMB00], there exists $q' \in R^{N,m}_{(v'_1,v'_2)}$ such that $R^{N,m}_{(v'_1,v'_2)} \ni q' \mapsto p' \in \overline{\mathscr{M}^{HN}_{(v_1,v_2)}(v)}$ and $q' \rightsquigarrow q$ in $R^{N,m}_{tf}$ and we think of q' as the generic point of $R^{N,m}_{(v'_1,v'_2)}$. So, we get $\dim \overline{R^{N,m}_{(v_1,v_2)}} = \dim R^{N,m}_{(v_1,v_2)}$ and $\dim \overline{R^{N,m}_{(v'_1,v'_2)}} = \dim R^{N,m}_{(v'_1,v'_2)}$. On the other hand, we have $\mathscr{M}^{HN}_{(v_1,v_2)}(v) \neq \mathscr{M}^{HN}_{(v'_1,v'_2)}(v)$ because $\overline{\mathscr{M}^{HN}_{(v_1,v_2)}(v)} \subseteq \overline{\mathscr{M}^{HN}_{(v'_1,v'_2)}(v)}$. Therefore, it holds that $\dim R^{N,m}_{(v'_1,v'_2)} > \dim R^{N,m}_{(v_1,v_2)}$. (if $\dim R^{N,m}_{(v'_1,v'_2)} = \dim R^{N,m}_{(v_1,v_2)}$, we have $\overline{R^{N,m}_{(v'_1,v'_2)}} = \overline{R^{N,m}_{(v_1,v_2)}}$, this contradicts to uniqueness of generic points.) $\qquad \square$

The next lemma is important to decide the irreducible components.

Lemma 3.13. *Let $v_1, v_2, v'_1, v'_2 \in \mathbb{Z} \bigoplus \mathrm{Pic}(X) \bigoplus \mathbb{Z}$. If $(v_1, v_2) \neq (v'_1, v'_2)$ i.e., $v_1 \neq v'_1$ or $v_2 \neq v'_2$, then $\overline{\mathscr{M}^{HN}_{(v_1,v_2)}(v)} \nsubseteq \overline{\mathscr{M}^{HN}_{(v'_1,v'_2)}(v)}$.*

Proof. If $E \in \mathscr{M}^{HN}_{(v_1,v_2)}(v)$ and the exact sequence

$$0 \longrightarrow \mathscr{I}_{Z_1}(m) \longrightarrow E \longrightarrow \mathscr{I}_{Z_2}(n - m) \longrightarrow 0$$

is the HN-filtration of E (where, $v(I_{Z_1}(m)) = v_1$, $v(I_{Z_2}(n-m)) = v_2$), then

$$\dim \mathscr{M}^{\mathrm{HN}}_{(v_1,v_2)}(v) = \langle v_1, v_1 \rangle + \langle v_2, v_2 \rangle + \langle v_1, v_2 \rangle + 2 = H^2(m - \frac{n}{2})^2 + (3c_2 - 4 - \frac{3n^2 H^2}{4}).$$

Under the notation of Lemma 3.12, the map $p \mapsto \mathrm{HNP}(p)$ is upper semicontinuous by [Sha77] or [Nit11]. So, we have $\mathrm{HNP}(p) \geq \mathrm{HNP}(p')$ and $\dim R^{N,m}_{(v_1,v_2)} \geq \dim R^{N,m}_{(v_1',v_2')}$ because $p' \rightsquigarrow p$. This is a contradiction. Therefore we get the proposition. □

3.3. **The proof of Theorem 1.1.** We prove Theorem 1.1 by using the lemmas before.

Proof. Any Mukai vector v satisfies one of the following disjoint conditions ;

$(a) : \langle v, v \rangle > 0,$ $\qquad\qquad\qquad\qquad\qquad (b) : \langle v, v \rangle = 0, -2$ and v is primitive ,

$(c) : \langle v_0, v_0 \rangle = 0, -2$ and v is non-primitive , $\qquad (d) : \langle v, v \rangle < -2$ and $\langle v_0, v_0 \rangle \neq -2.$

And, we will prove Theorem 1.1 in each case. We first have a stratification of $\mathscr{M}^{\mathrm{tf}}(v)$ by $\mathscr{M}^{\mathrm{ss}}(v)$ and $\mathscr{M}^{\mathrm{HN}}_{(v_1,v_2)}(v)$. (About HN stratification, for example, see [Nit11] or [Hos18], Section 5). In the case of (a) and (b), if $\langle v_1, v_2 \rangle \leq 1$,

$$\dim \mathscr{M}^{\mathrm{ss}}(v) = \langle v, v \rangle + 1 = \langle v_1, v_1 \rangle + \langle v_2, v_2 \rangle + 2\langle v_1, v_2 \rangle + 1 = \dim \mathscr{M}^{\mathrm{HN}}_{(v_1,v_2)}(v) + \langle v_1, v_2 \rangle - 1.$$

In the same way as in the proof of Lemma 3.13, we get $\overline{\mathscr{M}^{\mathrm{ss}}(v)} \not\supseteq \overline{\mathscr{M}^{\mathrm{HN}}_{(v_1,v_2)}(v)}$ and $\overline{\mathscr{M}^{\mathrm{ss}}(v)} \not\subseteq \overline{\mathscr{M}^{\mathrm{HN}}_{(v_1,v_2)}(v)}$. On the other hand, we consider the case $\langle v_1, v_2 \rangle > 1$. We assume $\overline{\mathscr{M}^{\mathrm{ss}}(v)} \not\supseteq \overline{\mathscr{M}^{\mathrm{HN}}_{(v_1,v_2)}(v)}$. Then, for genral $x \in \overline{\mathscr{M}^{\mathrm{HN}}_{(v_1,v_2)}(v)}$, we have $\dim_x \mathscr{M}^{\mathrm{tf}}(v) < \langle v, v \rangle + 1$ and this contradicts Remark 3.11. So we have $\overline{\mathscr{M}^{\mathrm{ss}}(v)} \supseteq \overline{\mathscr{M}^{\mathrm{HN}}_{(v_1,v_2)}(v)}$. By Lemma 3.4, Remark 3.5, and Corollary 3.7, the stacks $\mathscr{M}^{\mathrm{ss}}(v)$ and $\mathscr{M}^{\mathrm{HN}}_{(v_1,v_2)}(v)$ are irreducible. Therefore by Lemma 3.13, we can classify the irreducible components of $\mathscr{M}^{\mathrm{tf}}(v)$ as the statements of the theorem. In the case of (c), note that for any stack of HN-filtration, $\dim \mathscr{M}^{\mathrm{HN}}_{(v_1,v_2)}(v) > \dim \mathscr{M}^{\mathrm{ss}}(v)$ and $\langle v_1, v_2 \rangle \leq 0$. In the case (d), we have $\mathscr{M}^{\mathrm{ss}}(v) = \emptyset$ by [Yos99a, Cor 0.3]. So, we can classify the irreducible components. □

4. An application to Brill-Noether theory of Hilbert schemes of points

In [Wal95], an application of the irreducible components of moduli stacks of torsion free sheaves on ruled surfaces are performed. In this section, we replace ruled surfaces to K3 surfaces. For a K3 surface X, let N be a non-negative integer and let D be an effective divisor on X such that $h^0(X, \mathscr{O}(D)) \geq N$. And let $\mathrm{Hilb}^N(X)$ be the Hilbert scheme of finite schemes of length N on X. For the Hilbert schemes $\mathrm{Hilb}^N(X)$ of finite schemes of length N on X, We define $W_N^i(D)$ as follows.

$$W_N^i(D) := \{[Z] \in \mathrm{Hilb}^N(X) \mid h^1(\mathscr{I}_Z(D)) \geq i + 1\}.$$

Then, it is known that $W_N^i(D) \subseteq \mathrm{Hilb}^N(X)$ is a closed subscheme from upper semicontinuity of cohomology of flat families of sheaves and $h^1(\mathscr{I}_Z(D)) = i + 1$ for general members of each irreducible component of $W_N^i(D)$. In particular, if $i = 0$, we have a bijection between the irreducible components of $W_N^i(D)$ and the irreducible components of $\mathscr{M}^{\mathrm{tf}}(v)$ whose general member E satisfies the conditions $(1) : H^1(X, E) = H^2(X, E) = 0$ and $(2) : \exists s \in H^0(X, E)$ such that $E/s\mathscr{O}_X$ is torsion free. where, $v := (2, D, \frac{D^2}{2} - N + 2)$. Note that the conditions (1) and (2) are open conditions. Moreover, if E is a general member of an irreducible components $\mathscr{M}'$ of $\mathscr{M}^{\mathrm{tf}}(v)$ which satisfies $(1), (2)$ and let the corresponding irreducible component of $W_N^i(D)$ be V, then

$$(\spadesuit) \qquad\qquad\qquad\qquad \dim V = \dim \mathscr{M}' + h^0(E).$$

Proof of Theorem 1.3. We get the claim of Theorem 1.3 by the above comment, Lemma 4.1, Lemma 4.5 and calculating and rearranging $\chi(v) > 0$ and $h^0(\mathscr{O}_X(n-m)) > \ell_2$. □

4.1. **About not semistable components.**

Lemma 4.1. *Let* $v := (2, nH, \frac{n^2 H^2}{2} - N + 2)$, *let* $v_1 := (1, mH, \frac{m^2 H^2}{2} - \ell_1 + 1)$, *and let* $v_2 :=$ $(1, (n-m)H, \frac{(n-m)^2 H^2}{2} - \ell_2 + 1)$ *such that* $v = v_1 + v_2$. *We assume that* E *is a general member of* $\mathscr{M}^{HN}_{(v_1, v_2)}(v)$. *Then, E satisfies the conditions* (1), (2) *if and only if the following conditions hold.* $(a) : 2m \geq n > m > 0$, $(b) : \chi(E) > 0$, $(c) : h^0(\mathscr{O}_X(n-m)) > \ell_2$.

Proof. If the conditions (1), (2) are satisfied, we can easily prove that E satitsfy (a), (b) and (c). Note that the condition $h^0(\mathscr{O}(n-m)) = \ell_2$ can not occur. If $h^0(\mathscr{O}(n-m)) = \ell_2$, then all global section of general sheaf $E \in \mathscr{M}^{HN}_{(v_1, v_2)}(v)$ are included in $H^0(\mathscr{I}_{Z_1}(m))$ and non-zero sections in $H^0(\mathscr{I}_{Z_1}(m))$ never induce torsion free quotient. This contradicts to the condition (2).

Conversely, we assume that the conditions (a), (b) and (c) are satisfied. First, we prove the condition (1) by induction for ℓ_1 (cf. [Wal95, Lem 3.3 and Lem 4.5]). Note that $H^2(E) = 0$ for a general $E \in \mathscr{M}^{HN}_{(v_1, v_2)}(v)$ because $H^2(\mathscr{I}_{Z_1}(m)) = H^2(\mathscr{I}_{Z_2}(m)) = 0$. If $\ell_1 = 0$, then we have $H^1(E) = 0$ in the same way. For general $\ell_1 > 0$, we prove $H^1(E) = 0$ for a general $E \in \mathscr{M}^{HN}_{(v_1, v_2)}(v)$. We assume E' fits in the exact sequence

$$0 \to \mathscr{I}_{Z_1'}(m) \to E' \to \mathscr{I}_{Z_2'}(n-m) \to 0$$

which is the HN filtration of E' with $\ell(Z_1') = \ell_1 - 1$ and $\ell(Z_2') = \ell_2$. If $H^1(E') = 0$ and E' satisfies the conditions (a), (b) and (c), then, E' have a nonzero global section s. And, for a general point $x \in X$ and a general one dimensional quotient $E' \twoheadrightarrow E' \otimes k(x) \twoheadrightarrow k(x)$ of the fiber of E' at x denoted by φ, we have $\varphi(s) \neq 0$. Note that we can assume $x \notin Z_1'$. Let E be the kernel of φ. Then, we have $h^0(E) = h^0(E') - 1$ and $H^1(E) = 0$. And, we get the HN filtration of E

$$0 \to \mathscr{I}_{Z_1' \cup \{x\}}(m) \to E \to \mathscr{I}_{Z_2'}(n-m) \to 0$$

because of the HN-filtration of E' and the assumption $x \notin Z_1'$. So, we get condition (1) for general $\ell_1 > 0$.

Next, we prove the condition (2) under the condition (1).

We consider the conditions $(\alpha) : 2m = n$, $(\beta) : \ell_2 = 1$.

First, we suppose (α) and (β) are true. In this case, note that $\ell_1 = 0$ and every sheaf $E \in \mathscr{M}^{HN}_{(v_1, v_2)}(v)$ is isomorphic to $\mathscr{O}_X(m) \oplus \mathscr{I}_x(m)$ for some $x \in X$. We take section $s_1, s_2 \in H^0(\mathscr{O}(m))$ such that $Z(s_1) \cap Z(s_2)$ is a finite set, where $Z(s_i)$ is the zero set of $s_i (i = 1, 2)$. If $x \in Z(s_2)$, $s_1 \oplus s_2 \in H^0(\mathscr{O}_X(m) \oplus \mathscr{I}_x(m))$. Since $s_1 \oplus s_2 \in \mathscr{O}_X(m) \oplus \mathscr{O}_X(m)$ defines a torsion free quotient $\mathscr{O}_X(m) \oplus \mathscr{O}_X(m)/(s_1 \oplus s_2)\mathscr{O}_X$, $\mathscr{O}_X(m) \oplus \mathscr{I}_x(m)/(s_1 \oplus s_2)\mathscr{O}_X$ is also torsion free.

When (α) or (β) are not true, it is sufficient to prove the following.

Claim 4.2. *Let k be a positive integer. We assume that $\ell_1 = 0$. Then, we have*

$$h^0(E(-k)) + \dim |kH| < h^0(E).$$

If the claim is true, then a general $E \in \mathscr{M}^{HN}_{(v_1, v_2)}(v)$ with $\ell_1 = 0$ is a vector bundle because the Cayley-Bacharach property holds for a pair a general Z_2 and $\mathscr{O}_X(n - 2m)$ by the choice of m and ℓ_2 (cf. [HL10, Thm 5.1.1]). And, the set $H^0(E) \setminus \bigcup_{C \in |kH|, k \in \mathbb{N}} H^0(E(-C))$ is a non-empty open set from Claim 4.2. So, a general section s of a general $E \in \mathscr{M}^{HN}_{(v_1, v_2)}(v)$ with $\ell_1 = 0$ defines a torsion free quotient $E/s\mathscr{O}_X$ because the zero set $Z(s)$ of s is a finite set (cf. [OSS11, Ch. 1, §5]).

In the case $\ell_1 > 0$, we have a torsion free sheaf E' fitting into the sequence

$$0 \to \mathscr{O}_X(m) \to E' \to \mathscr{I}_{Z_2}(n-m) \to 0$$

whose general section s determines a torsion free quotient because of the case $l_1 = 0$ and the case (α) and (β) are true.

In addition, E' is generically globally generated. Note that we say that E' is generically generated if the evaluation map $\mathrm{ev} : H^0(E') \otimes \mathscr{O}_X \to E'$ is surjective on an open set of X. Actually, from the condition (a) and (c), $\mathscr{O}_X(m)$ and $\mathscr{I}_{Z_2}(n-m)$ is generically globally generated. So, a simple diagram chase shows that E' is generically global generated.

Let U is the subset of $H^0(E')$ of the sections defining torsion free quotient. Then, the natural $\mathbb{C}$-linaer homomorphism $\widetilde{\mathrm{ev}} : H^0(E') \to E' \otimes k(x)$ is surjective and $\widetilde{\mathrm{ev}}|_U$ is dominant for general $x \in X$ because E' is generically globally generated. So, we can take general ℓ_1 points $x_1, \cdots, x_{\ell_1}$ on X and a general sections s such that $s \notin \mathscr{O}_X(m) \otimes k(x_i)$ for all i and s defines a torsion free quotient. Then, we take one-dimensional quotients $\varphi_i : E' \twoheadrightarrow E' \otimes k(x_i) \twoheadrightarrow k(x_i)(1 \le i \le \ell_1)$ such that $\varphi_i|_{\mathscr{O}_X(m)} \ne 0$ for all i and $\varphi_i(s) = 0$ for all i. We consider the quotient $\varphi : E' \twoheadrightarrow \bigoplus_{i=1}^{\ell_1} k(x_i)$ obtained from φ_i. If let E be the kernel of φ, $E \in \mathscr{M}^{\mathrm{HN}}_{(v_1,v_2)}(v)$ and a general section s of E defines a torsion free quotient.

Proof of Claim 4.2. In this paper, we only consider the case

$$2m > n > m > 0, \qquad \ell(Z_2) > h^0(\mathscr{O}(n-m-1) = \frac{(n-m-1)^2}{2}H^2 + 2,$$

$$m \ge 3, \qquad n - m \ge 3.$$

The other cases can be proved in the same way or more easily.

Because $h^0(\mathscr{O}(n-m-1) < \ell(Z_2) < h^0(\mathscr{O}(n-m))$, we have $H^0(\mathscr{I}_{Z_2}(n-m-k)) = 0$ for all positive integer k and general Z. So, we have $H^0(E(-k)) = H^0(\mathscr{O}(m-k))$.

In this condition, we have $H^0(E(-k)) = \chi(\mathscr{O}(m-k)) = \frac{(m-k)^2}{2}H^2 + 2$ and $\dim|kH| = h^0(\mathscr{O}(kH)) - 1 = \frac{k^2}{2}H^2 + 1$. Note that $h^0(E) = \chi(E) = \frac{m^2}{2}H^2 + \frac{(n-m)^2}{2}H^2 + 4 - \ell(Z_2)$. Then, we can calculate as follows.

$$h^0(E) - \{h^0(E(-k)) + \dim|kH|\} = \frac{m^2}{2}H^2 + \frac{(n-m)^2}{2}H^2 - \frac{(m-k)^2}{2}H^2 - \frac{k^2}{2}H^2 + 1 - \ell(Z_2).$$

In addition, we have $\ell(Z_2) < h^0(\mathscr{O}(n-m)) = \frac{(n-m)^2}{2}H^2 + 2$. So,

$$h^0(E) - \{h^0(E(-k)) + \dim|kH|\} > H^2 k(m-k) - 1 > 0 (\because k, m-k > 0).$$

$\square$

Thus, we get the condition (2) when $2m \ne n$ or $\ell_2 \ne 1$, and we complete the proof of Lemma. $\square$

4.2. **About semistable components.** We will use the following lemmas to prove the Lemma 4.5.

Lemma 4.3 ([Yos99a] Lemma 1.4 or [Yos99b] Lemma 2.1). *Let n be an odd integer. If the exact sequence*

$$0 \to \mathscr{O}\left(\frac{n-1}{2}\right) \to E \to \mathscr{I}_Z\left(\frac{n+1}{2}\right) \to 0$$

does not split, then E is a μ-stable sheaf, where $\mathscr{I}_Z$ is the ideal sheaf of a finite subscheme Z.

Lemma 4.4 ([Yos99a] Proposition 0.5 and Section 3.3). *Let $v := (2, nH, \frac{n^2 H^2}{2} - N + 2)$. We assume that v is primitive, $v \ne (2, 0, -1)$ and " $v \ne (2, nH, \frac{n^2 H^2}{4} - 1)$ and n is even". Then, there exists a stable locally free sheaf.*

Lemma 4.5. *Let E be a general member of the stack $\mathscr{M}^{ss}(v)$. Then, the conditions (1) and (2) are equal to the conditions $\chi(E) > 0$ and " $H^2 \ne 2, v \ne (2, 3H, 5)$ ".*

Proof. If (1) and (2) satisfy, we have $H^1(E) = 0$ and $H^0(E) \ne 0$. Therefore, we have $\chi(E) > 0$.

We will prove Lemma 4.5 only when n is an odd integer. We can also prove this lemma in the same way when n is even. Note, for a general E, we have $H^2(E) = 0$ by semistability.

First, we assume that $N > \frac{n^2+1}{4}H^2 + 3$. This is equivalent to the condition that the closure of the stacks of Harder-Narasimhan filtrations whose general sheaf is an extension

$$0 \to \mathscr{I}_{Z_1}\left(\frac{n+1}{2}\right) \to E \to \mathscr{I}_{Z_2}\left(\frac{n-1}{2}\right) \to 0$$

is contained in the closure of $\mathscr{M}^{ss}(v)$. Then, we can show that some E in the closure of $\mathscr{M}^{ss}(v)$ have no higher cohomology in the same way as in Lemma 4.1. Moreover, we can prove a general E have a global section which give a torsion free quotient.

Next we assume that $\frac{n^2+1}{4}H^2 + 3 \geq N$. From Lemma 4.4, there exists a μ-stable sheaf E. We next consider $E(-\frac{n-1}{2})$. Let $E' := E(-\frac{n-1}{2})$ and $v' := v(E')$. Then, E' fits into the following exact sequence

$$0 \to \mathscr{O} \to E' \to \mathscr{I}_Z(1) \to 0$$

, where Z is a finite subscheme of X. Indeed, we have $\hom(E'^\vee, \mathscr{O}) = \operatorname{ext}^2(\mathscr{O}, E'^\vee) = h^2(E'^\vee) = h^0(E') \neq 0$ and $\hom(E', \mathscr{O}) = \operatorname{ext}^2(\mathscr{O}, E') = h^2(E') = 0$ because $\chi(E') > 0$ and , E' is also μ-stable. So, we have the above exact sequence by using these.

Because any non-split extension of $\mathscr{I}_Z(1)$ by $\mathscr{O}$ is a μ-stable sheaf from Lemma 4.3, the unique irreducible component of $W^0_{\ell(Z)}(H)$ corresponds to $\mathscr{M}^{\mathrm{ss}}(v')$. Thus, $h^1(\mathscr{I}_Z(1)) = 1$, for a general E since $\mathscr{M}^{\mathrm{ss}}(v) \simeq \mathscr{M}^{\mathrm{ss}}(v')$. Thus, we prove that $h^1(\mathscr{I}_{Z'}(k+1)) = 0$ for general $Z' \in W^0_{\ell(Z)}(H)$ for any $k > 0$ and $h^1(\mathscr{I}_{Z'}(2)) \neq 0$ for all $Z' \in W^0_{\ell(Z)}(H)$ only in the case $H^2 = \ell(Z) = 2$. It is sufficient to prove that $h^1(\mathscr{I}_{Z'}(2)) = 0$ for general $Z' \in W^0_{\ell(Z)}(H)$ except for $H^2 = \ell(Z) = 2$ because $h^1(\mathscr{I}_{Z'}(k)) \geq h^1(\mathscr{I}_{Z'}(k+1))$. In the case $H^2 = \ell(Z) = 2$, we prove $W^0_{\ell(Z)}(H) = W^0_{\ell(Z)}(2H)$ and $h^1(\mathscr{I}_{Z'}(3)) = 0$ for general $Z' \in W^0_{\ell(Z)}(H)$.

First, note that we have $\frac{H^2}{2} + 3 \geq \ell(Z)$ because $\chi(E') > 0$. If $\frac{H^2}{2} + 3 = \ell(Z)$, then $\operatorname{Hilb}^{\ell(Z)}(X) = W^0_{\ell(Z)}(H)$. For general $Z' \in \operatorname{Hilb}^{\ell(Z)}(X)$, $h^1(\mathscr{I}_{Z'}(2)) = 0$ because $h^0(\mathscr{O}_X(2)) \geq \ell(Z)$. If $\frac{H^2}{2} + 2 \geq \ell(Z)$, then $\operatorname{Hilb}^{\ell(Z)}(X) \neq W^0_{\ell(Z)}(H)$. We assume that $H^1(\mathscr{I}_Z(2)) \neq 0$ for any $Z' \in W^0_{\ell(Z)}(H)$. This is equal to $W^0_{\ell(Z)}(H) = W^0_{\ell(Z)}(2H)$. Let $v'' := (2, 2H, 2H^2 - \ell(Z) + 2)$. Then, $\mathscr{M}^{\mathrm{ss}}(v'') = \emptyset$ unless " $H^2 = \ell(Z) = 4$ " or " $H^2 = \ell(Z) = 2$ " by [Yos99a, Cor 0.3]. Moreover, there is not an irreducible component of $\mathscr{M}^{\mathrm{tf}}(v'')$ whose general member is a HN-filtration satisfying the conditions (a), (b) and (c) of Lemma 4.1 unless " $H^2 = 2$ and $\ell(Z) = 3$ ". So, from Lemma 4.1, $W^0_{\ell(Z)}(2H) = \emptyset$ except the three cases. This contradicts to $W^0_{\ell(Z)}(H) \neq \emptyset$. When $H^2 = \ell(Z) = 4$, $W^0_{\ell(Z)}(2H)$ may not be empty. If $W^0_{\ell(Z)}(2H)$ is not empty, the unique irreducible component corresponds to $\mathscr{M}^{\mathrm{tf}}(v'')$. Moreover we can calculate the dimensions of $W^0_{\ell(Z)}(H)$ and $W^0_{\ell(Z)}(2H)$ by using 3.8 and the formula ♠ and get $\dim W^0_{\ell(Z)}(H) = 7$ and $\dim W^0_{\ell(Z)}(2H) = 4$. When $H^2 = \ell(Z) = 2$, $W^0_{\ell(Z)}(2H) \neq \emptyset$ because the unique point of $\mathscr{M}^{\mathrm{ss}}(v'')$ is $\mathscr{O}_X(H)^{\oplus 2}$. And, we also have $\dim W^0_{\ell(Z)}(H) = \dim W^0_{\ell(Z)}(2H) = 2$. So, we have $W^0_{\ell(Z)}(H) = W^0_{\ell(Z)}(2H)$. However, $W^0_{\ell(Z)}(3H) = \emptyset$ as above. In the same way, we get a contradiction when $H^2 = 2$ and $\ell(Z) = 3$. Therefore, we get $h^1(\mathscr{I}_{Z'}(2)) = 0$ for a general $Z' \in W^0_{\ell(Z)}(H)$ when $H^2 \neq 2$ or $\ell(Z) \neq 2$ and $h^1(\mathscr{I}_{Z'}(3)) = 0$ for a general $Z' \in W^0_{\ell(Z)}(H)$ when $H^2 = \ell(Z) = 2$.

Remark 4.6. We will explain how we calculate $\dim W^0_{\ell(Z)}(H)$ and $\dim W^0_{\ell(Z)}(H)$ when $H^2 = \ell(Z) = 4$ here (similarly, we can also do when " $H^2 = \ell(Z) = 2$ " and " $H^2 = 2, \ell(Z) = 3$ "). It is sufficient to calculate $\dim \mathscr{M}^{\mathrm{ss}}(v')$, $\dim \mathscr{M}^{\mathrm{ss}}(v'')$, $h^0(E')$ and $h^0(E'')$, where E'' is a general member of $\mathscr{M}^{\mathrm{ss}}(v'')$ from the formula ♠. We can calculate $\dim \mathscr{M}^{\mathrm{ss}}(v')$ and $\dim \mathscr{M}^{\mathrm{ss}}(v'')$ by using 3.8, $h^0(E')$ by using the above exact sequence and $h^0(E'')$ by the fact that the unique member of $\mathscr{M}^{\mathrm{ss}}(v'')$ is $\mathscr{O}_X(H)^{\oplus 2}$ (in detail, see [Muk84a], [KLS06] et al.).

Next,we prove the condition (2). It is enough to prove $h^0(E(-k)) + \dim |kH| < h^0(E)$ as in the same way of the proof of Lemma 4.1 because a general sheaf in $\mathscr{M}^{\mathrm{ss}}(v)$ is a vector bundle by Lemma 4.4. Note that we have the following exact sequence for a general E,

$$0 \to \mathscr{O}\left(\frac{n-1}{2}\right) \to E \to \mathscr{I}_Z\left(\frac{n+1}{2}\right) \to 0$$

, where Z is a finite subscheme of X and $h^1(\mathscr{I}_Z\left(\frac{n+1}{2}\right)) = 0$. So, for $\frac{n-1}{2} \geq k > 0$,

$$h^0(E) - \{h^0(E(-k)) + \dim |kH|\}$$
$$\geq (h^0(\mathscr{I}_Z(\frac{n+1}{2})) + \chi(\mathscr{O}(\frac{n-1}{2})) - (h^0(\mathscr{I}_Z(\frac{n+1}{2} - k)) + \chi(\mathscr{O}(\frac{n-1}{2} - k)) + \dim |kH|)$$
$$= kH^2(\frac{n-1}{2} - k) - 1 + h^0(\mathscr{I}_Z(\frac{n+1}{2})) - h^0(\mathscr{I}_Z(\frac{n+1}{2} - k)) > 0.$$

(In the case of $k = \frac{n-1}{2}$, we use $h^0(\mathscr{I}_Z(\frac{n+1}{2})) = \frac{(n+1)^2}{8}H^2 - \ell(Z) + 2$ and $h^0(\mathscr{I}_Z(1)) = \frac{1}{2}H^2 - \ell(Z) + 3$)

Remark 4.7. (In the case n is even) When n is even, we can prove that a general sheaf $E \in \mathscr{M}^{\mathrm{ss}}(v)$ have a section defining a torsion free quotient as in the same way as in the proof above except $v = (2, nH, \frac{n^2H^2}{2})$ or $(2, nH, \frac{n^2H^2}{4} - 1)$. In these case, any sheaf of $\mathscr{M}^{\mathrm{ss}}(v)$ is not vector bundle and the closure of $\mathscr{M}^{\mathrm{ss}}(v)$ dose not contain any stacks of HN-filtration. However, we can prove the condition (1), (2) in the same way of the proof of Lemma 4.1. In the former case, note that any semistable sheaf is isomorphic to a sheaf of the form $\mathscr{I}_x\left(\frac{n}{2}\right) \oplus \mathscr{I}_y\left(\frac{n}{2}\right)(x, y \in X)$. In the latter case, note that a general quotient $\mathscr{O}\left(\frac{n}{2}\right) \to \oplus_{i=1}^3 k(x_i)(x_i \in X)$ and any non split extension $0 \to \mathscr{I}_{\{y_1,y_2\}}\left(\frac{n}{2}\right) \to E \to \mathscr{I}_{y_3}\left(\frac{n}{2}\right) \to 0(y_j \in X)$ is a sheaf with the Mukai vector $v = (2, nH, \frac{n^2H^2}{4} - 1)$ when n is even (cf. [Yos99a, Prop 3.4]).

$\square$

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